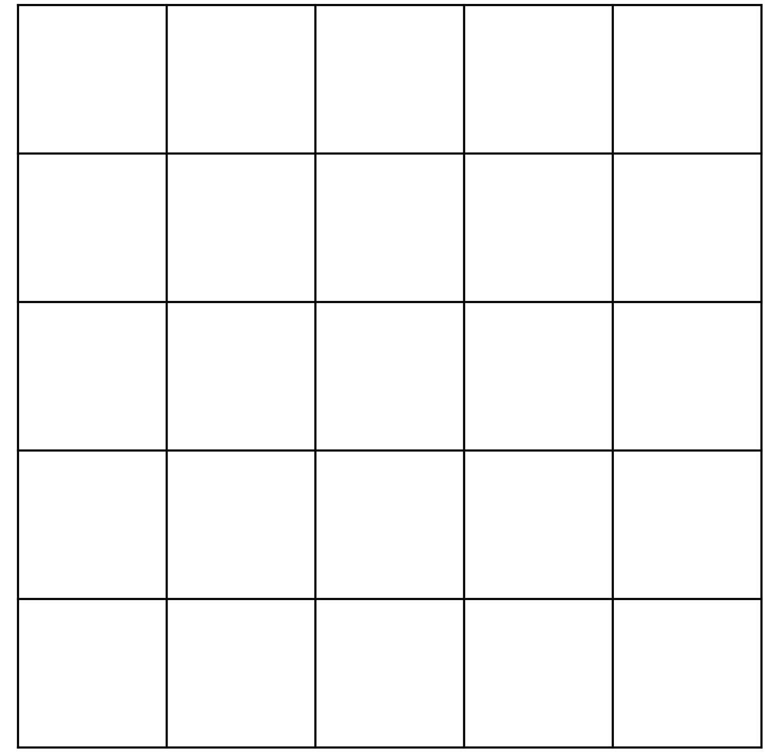
Planar graph embeddings and stat mech

Richard Kenyon (Brown University)

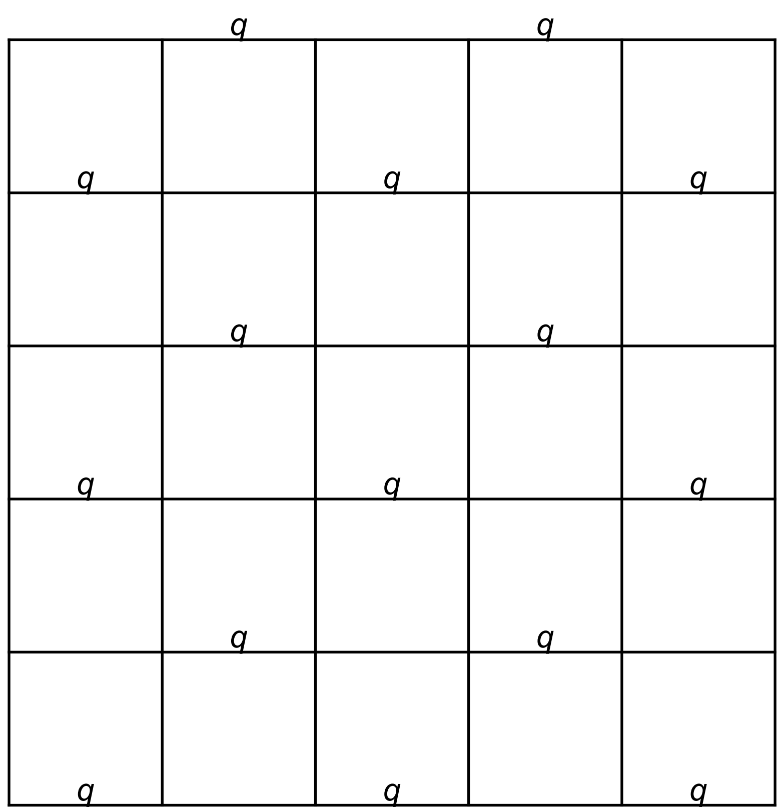
In 2D stat mech models, appropriate graph embeddings are important

e.g. Bond percolation on $\mathbb{Z}^2$.

$$p_c = \frac{1}{2}$$

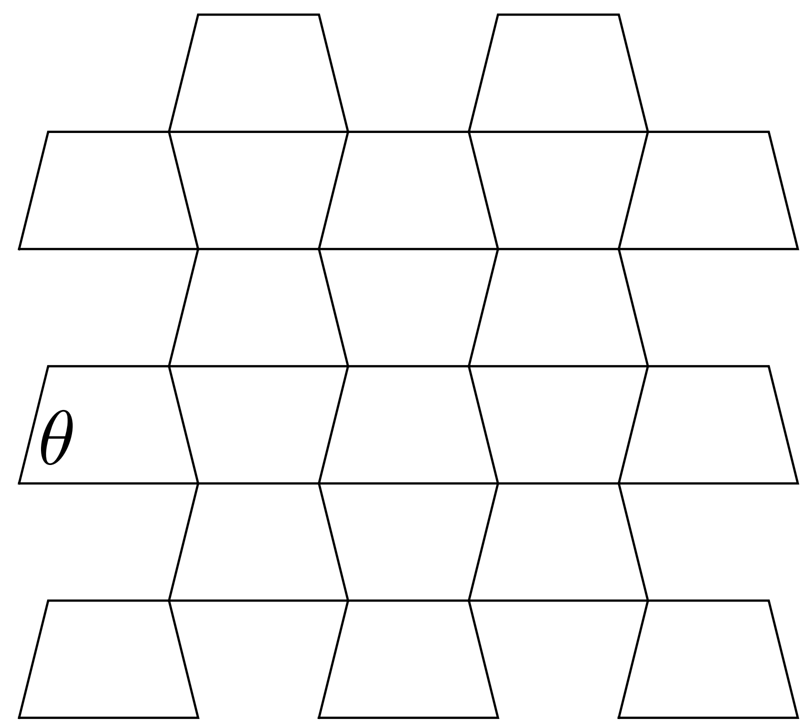


What about unequal probabilities?



critical if:

$$p^3 + 3p^2q - 3p^2 - 3pq + 1 = 0$$



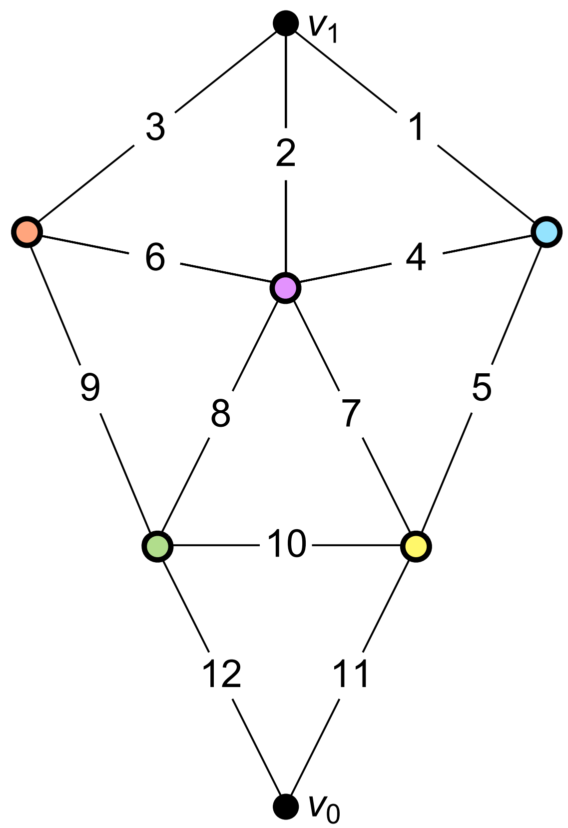
$$\theta = \theta(p,q)$$

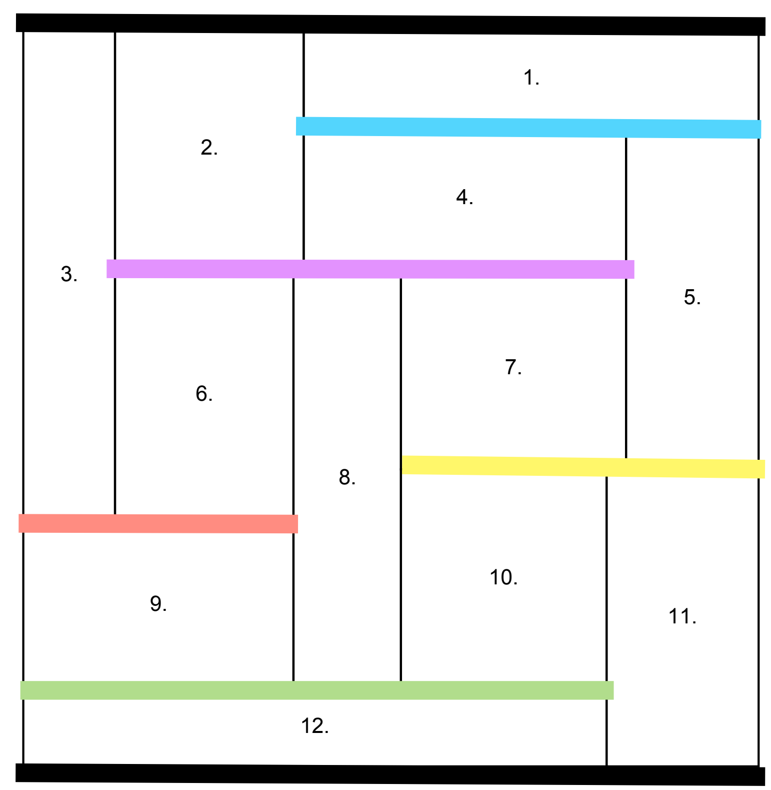
In 2D stat mech models, appropriate graph embeddings are important

Random walks/spanning trees BSST, LSW, Georgakopoulos Angel, Barlow, Gurel-Gurevich, Nachmias Hutchcroft, Peres	harmonic embedding, square tiling circle packing trapezoid tiling
Dimer models (Kenyon, Sheffield)	T-graphs
Ising model (Kenyon, Mercat, Smirnov)	K-graphs
FK (random cluster) model	isoradial graphs
bipolar orientations (Abrams, Kenyon)	area-1 rectangulations
Schnyder woods (Schnyder, , X. Sun, Watson)	Schnyder embedding
Random planar maps (KPZ, Duplantier, Miller, Sheffield)	conformal

1. T-graphs and dimers
2. Convex embeddings of a planar graph
3. Harmonic embeddings
4. Discrete analytic functions
5. Fixed-area rectangulations

Smith diagram of a planar network [BSST 1939]
(with a harmonic function)





vertex = horizontal line

voltage = y-coordinate

edge = rectangle

current = width

conductance = aspect ratio (width/height)

energy = area

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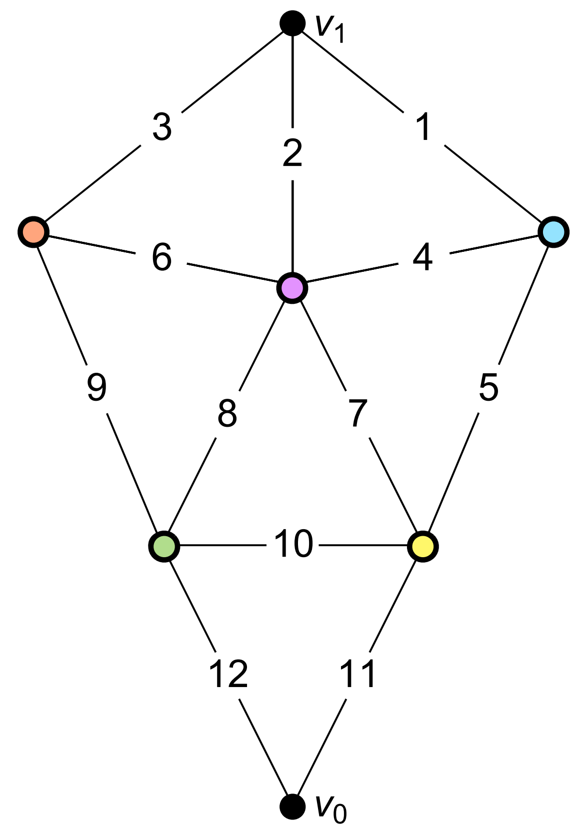

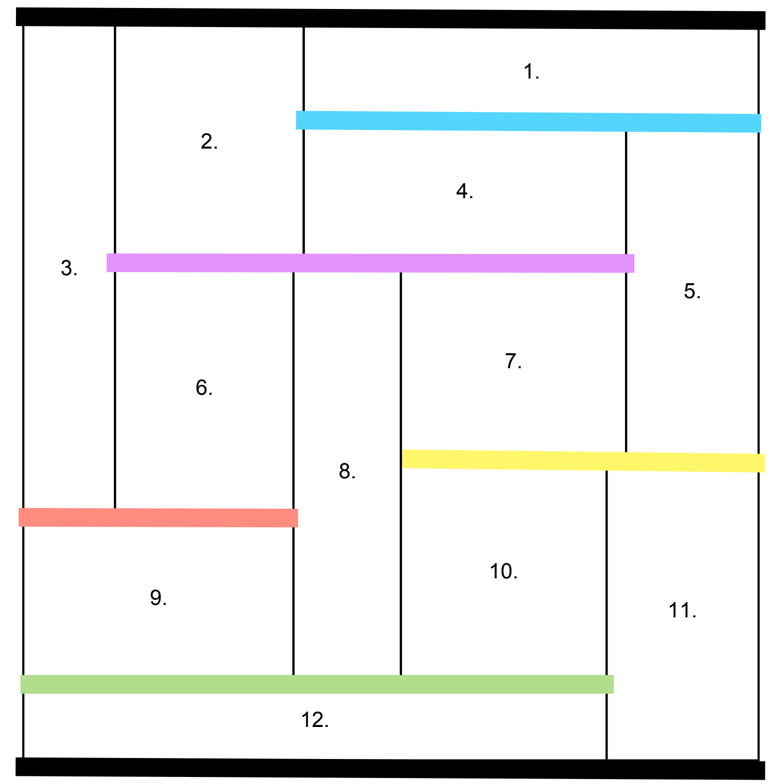

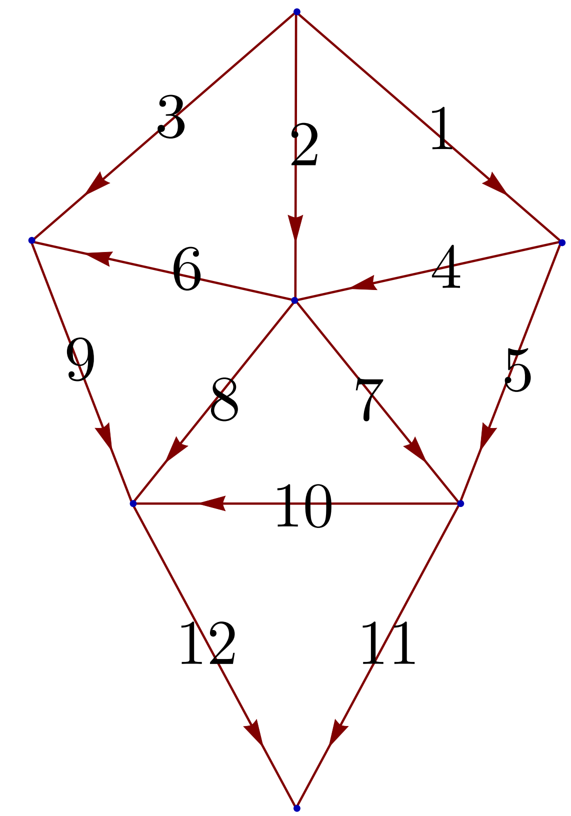

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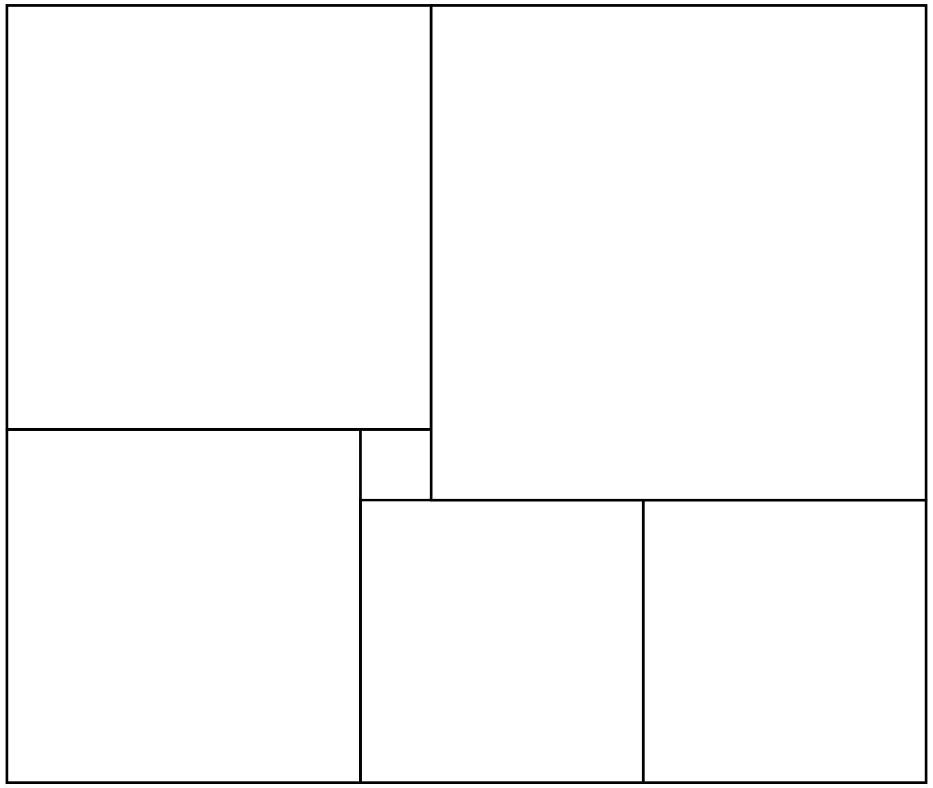
Thm(Dehn 1903): An $a \times b$ rectangle can be tiled with squares iff $a/b \in \mathbb{Q}$.



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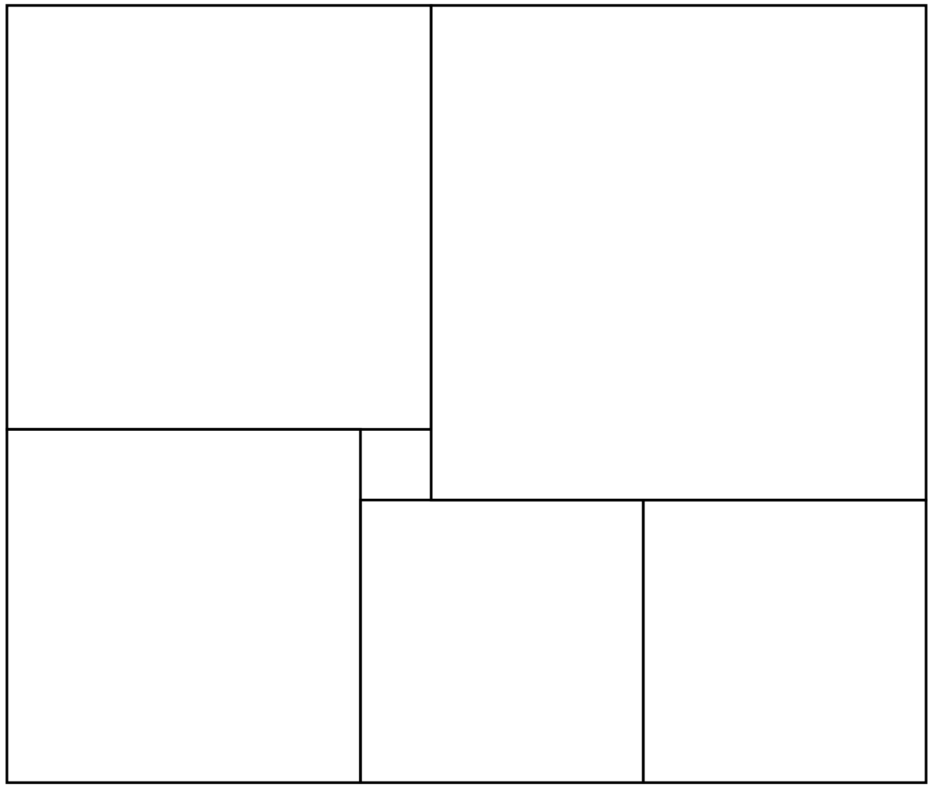
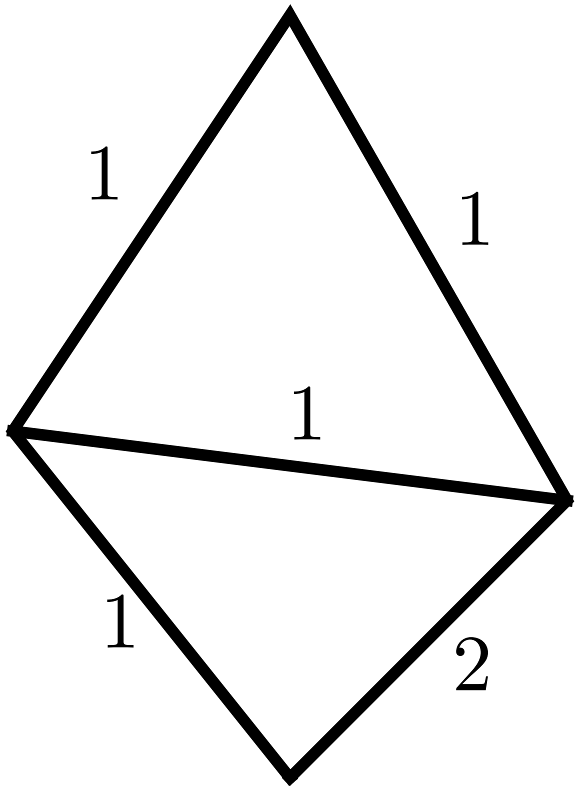
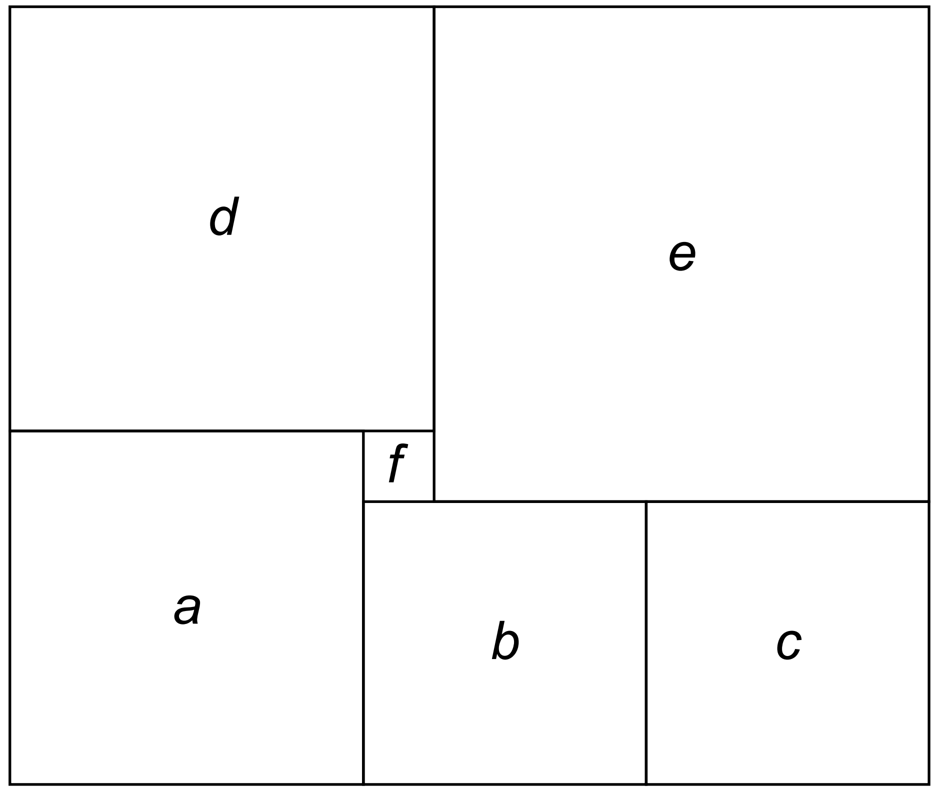




alternate proof

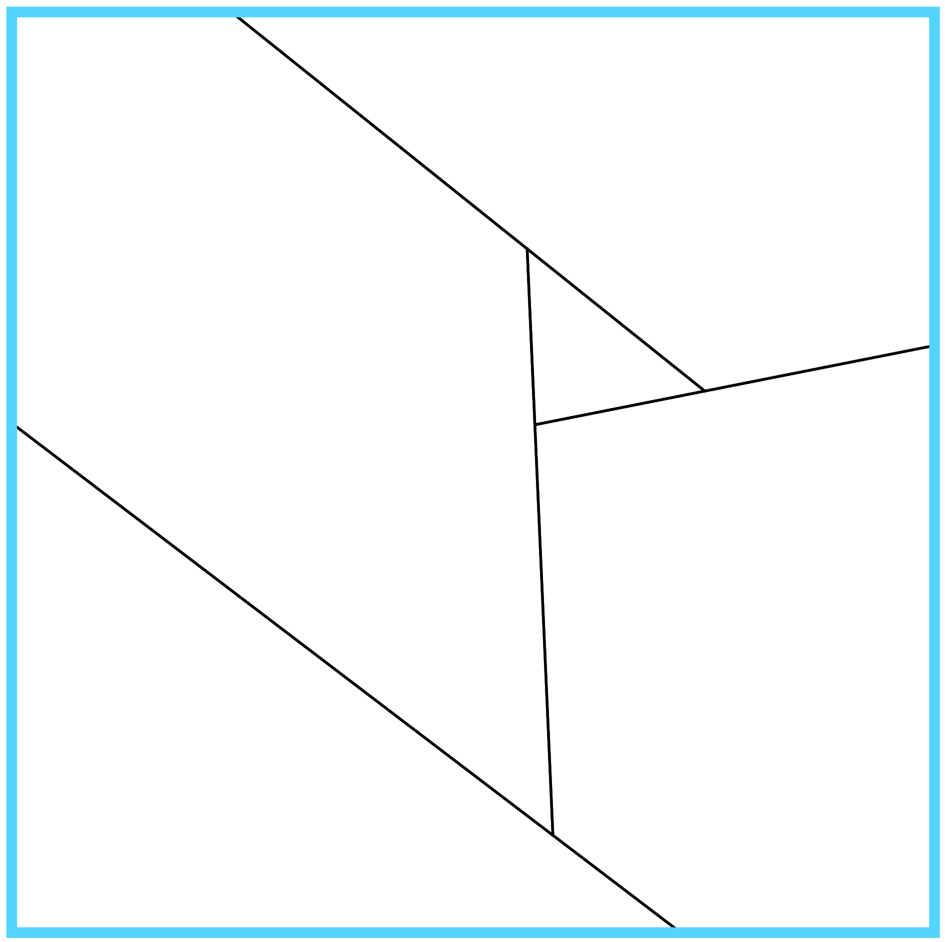


$$a + d = 1$$
$$a + f - d = 0$$
$$a - b - f = 0$$
$$d + f - e = 0$$
$$b + c - f - e = 0$$
$$b - c = 0$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \\ e \\ f \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \det K = ?$$

K is a signed adjacency matrix of an underlying planar graph...

A *t-graph in a polygon* is a union of noncrossing line segments in which every endpoint lies on another segment, or on the boundary, or at a point where three or more segments meet, with one in each halfspace.



a t-graph with four segments

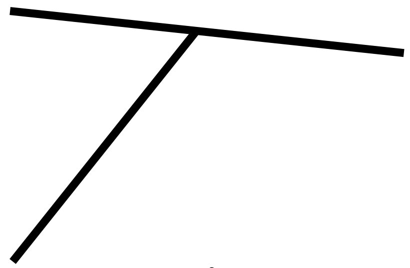
A t-graph is *generic* if no two endpoints are equal.

Note: faces are convex.

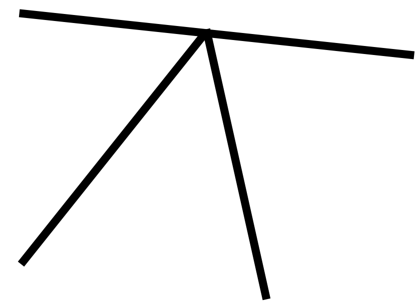
For generic t-graphs,

$$1 = \chi(\text{open disk}) = \#(\text{faces}) - \#(\text{segments}).$$

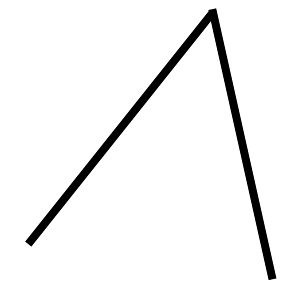
local pictures:



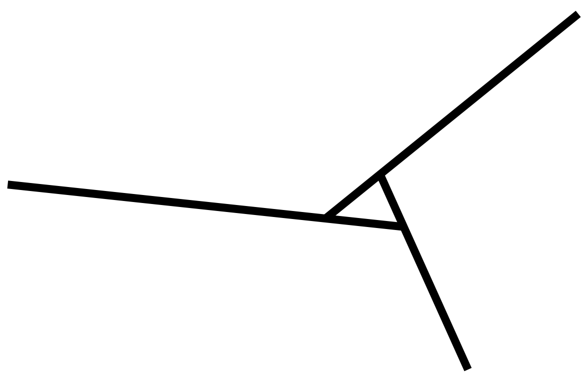
generic



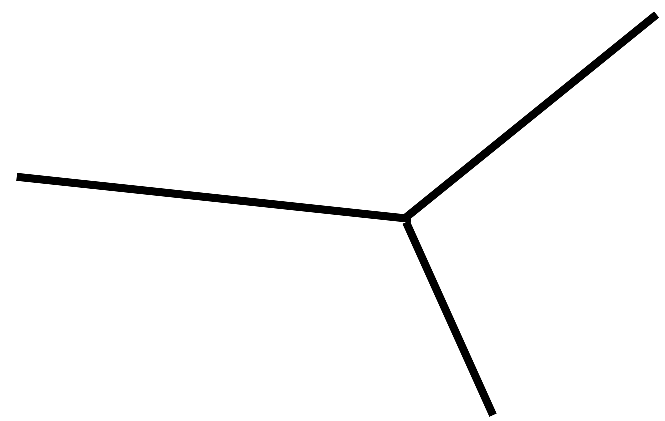
nongeneric



not allowed



generic



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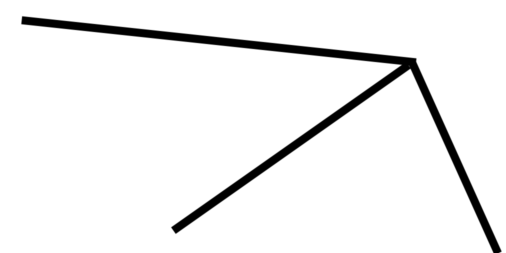

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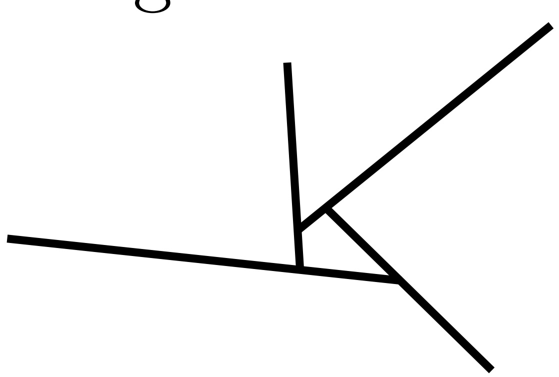
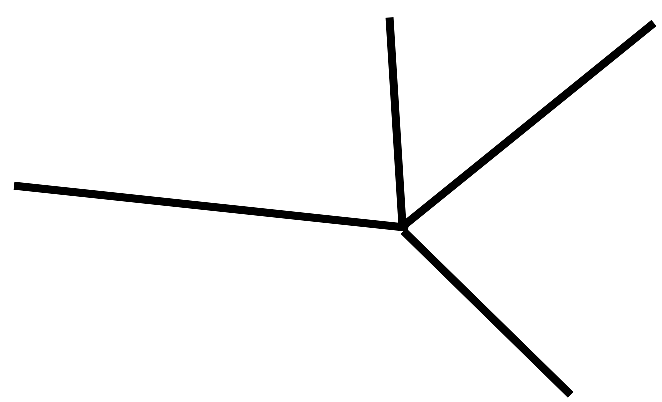


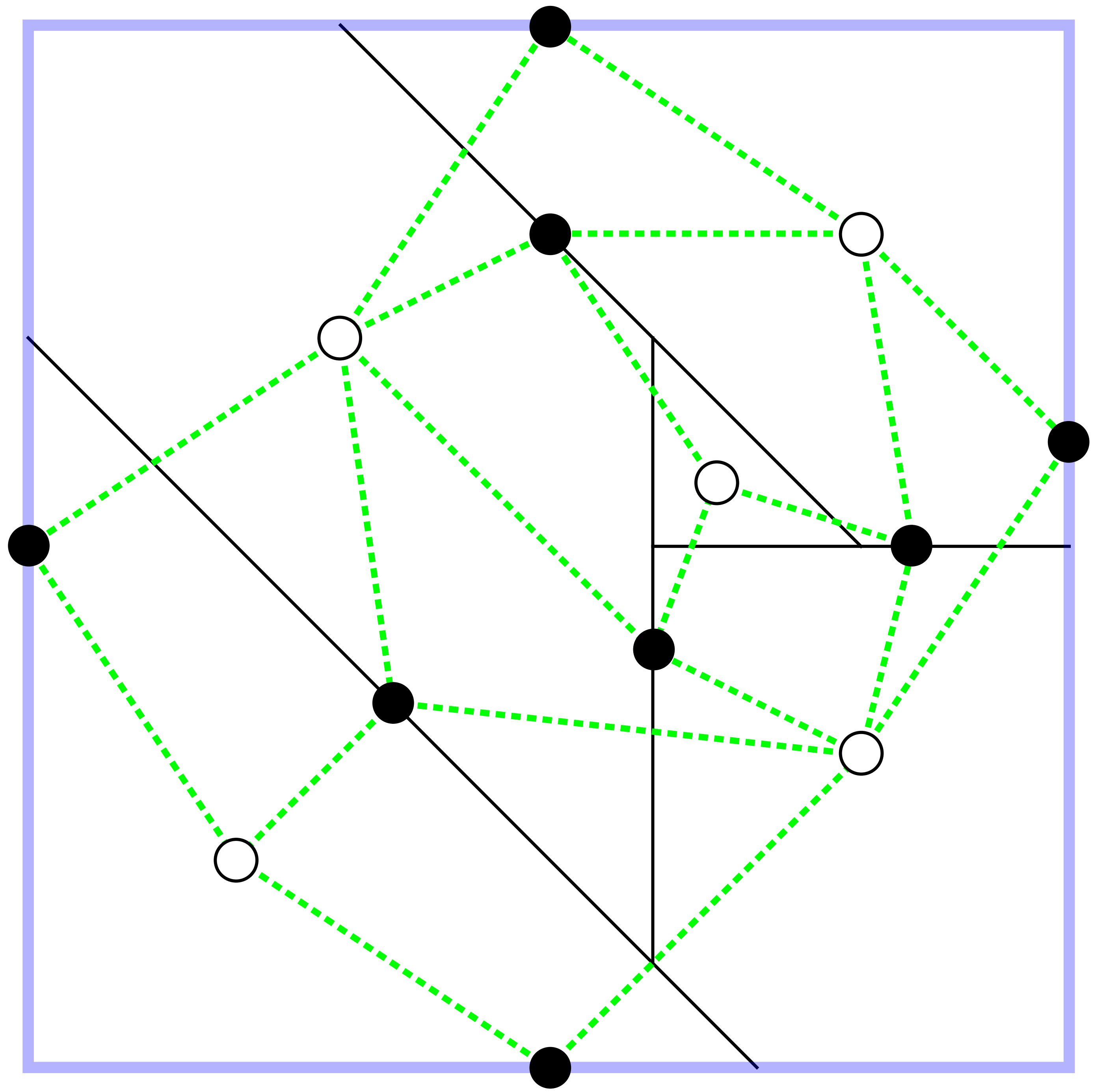
generic

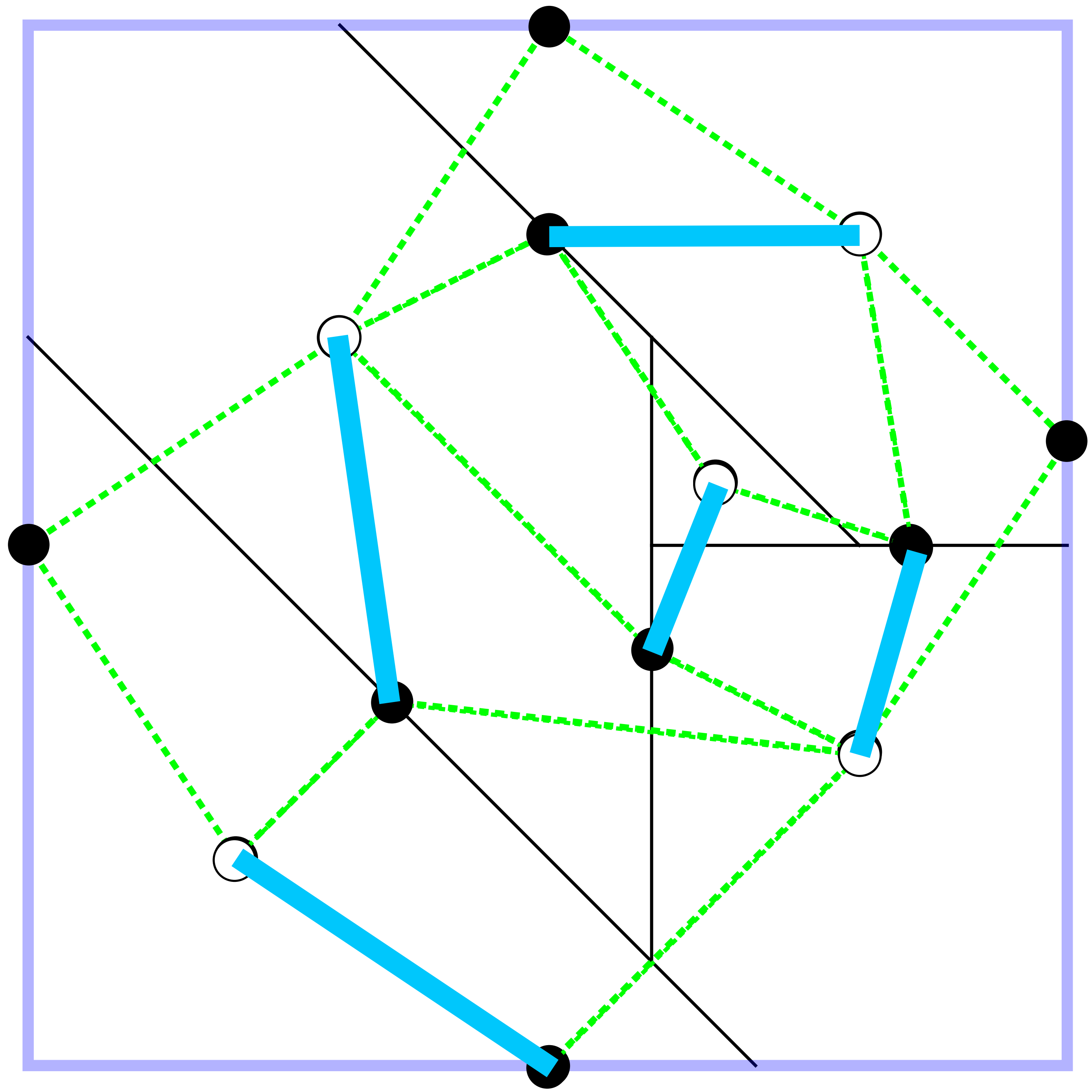


nongeneric

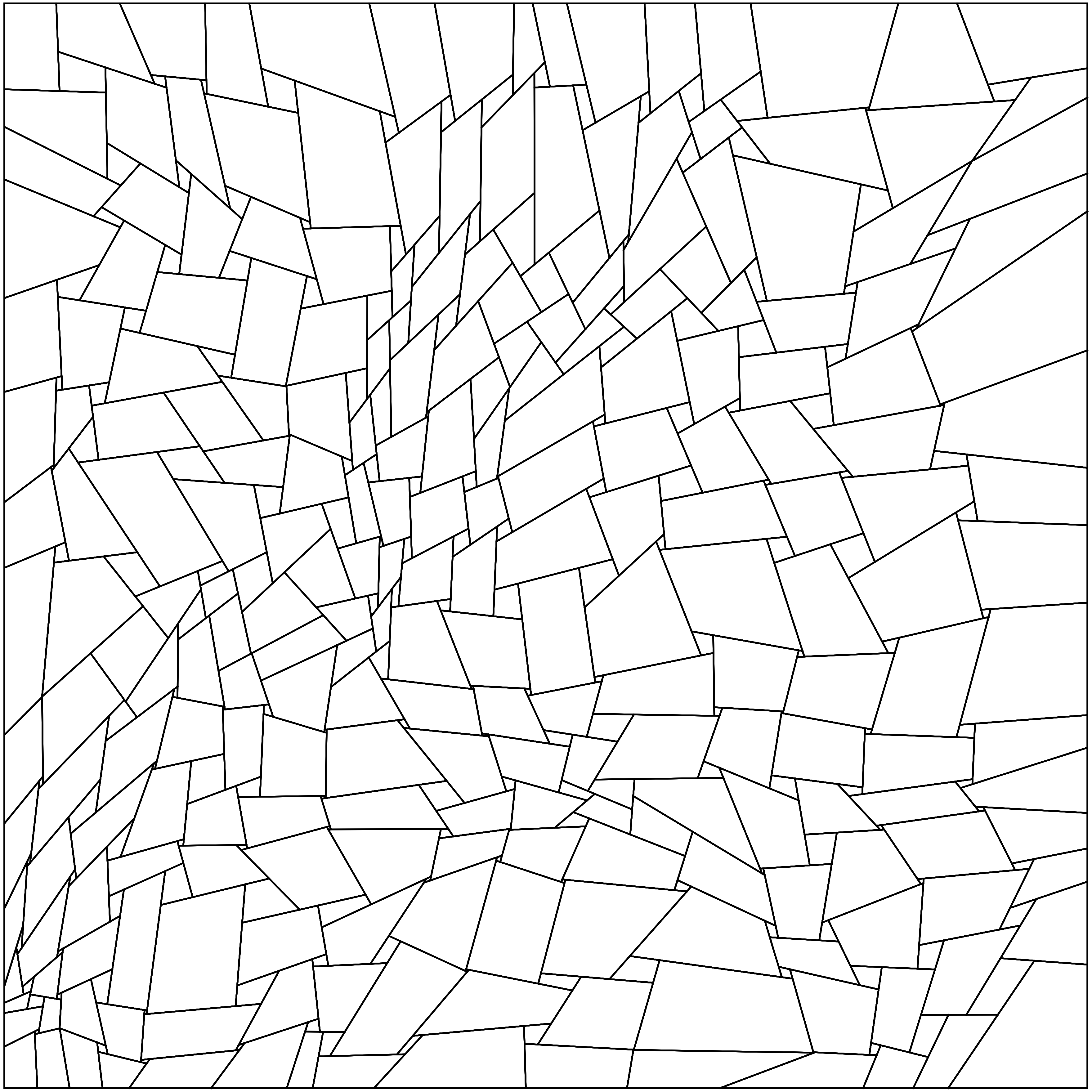
(if three or more segments meet at a point, there must be one in each halfspace)

Associated to a t-graph is a bipartite graph...





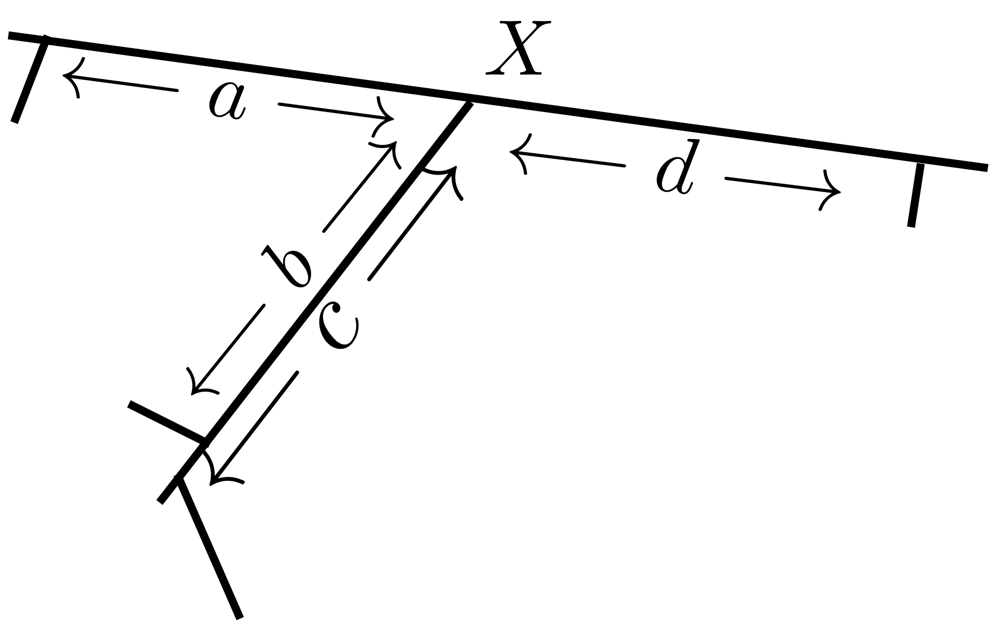
...which has dimer covers (when we remove all but one outer edge).



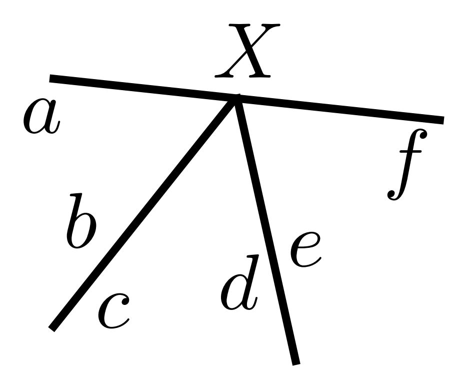
(follows from [K-Sheffield 2003])

Thm: The space of t-graphs with n segments, fixed boundary and fixed combinatorics is homeomorphic to $\mathbb{R}^{2n}$.

Global coordinates are *biratio coordinates* $\{X_i\}$.

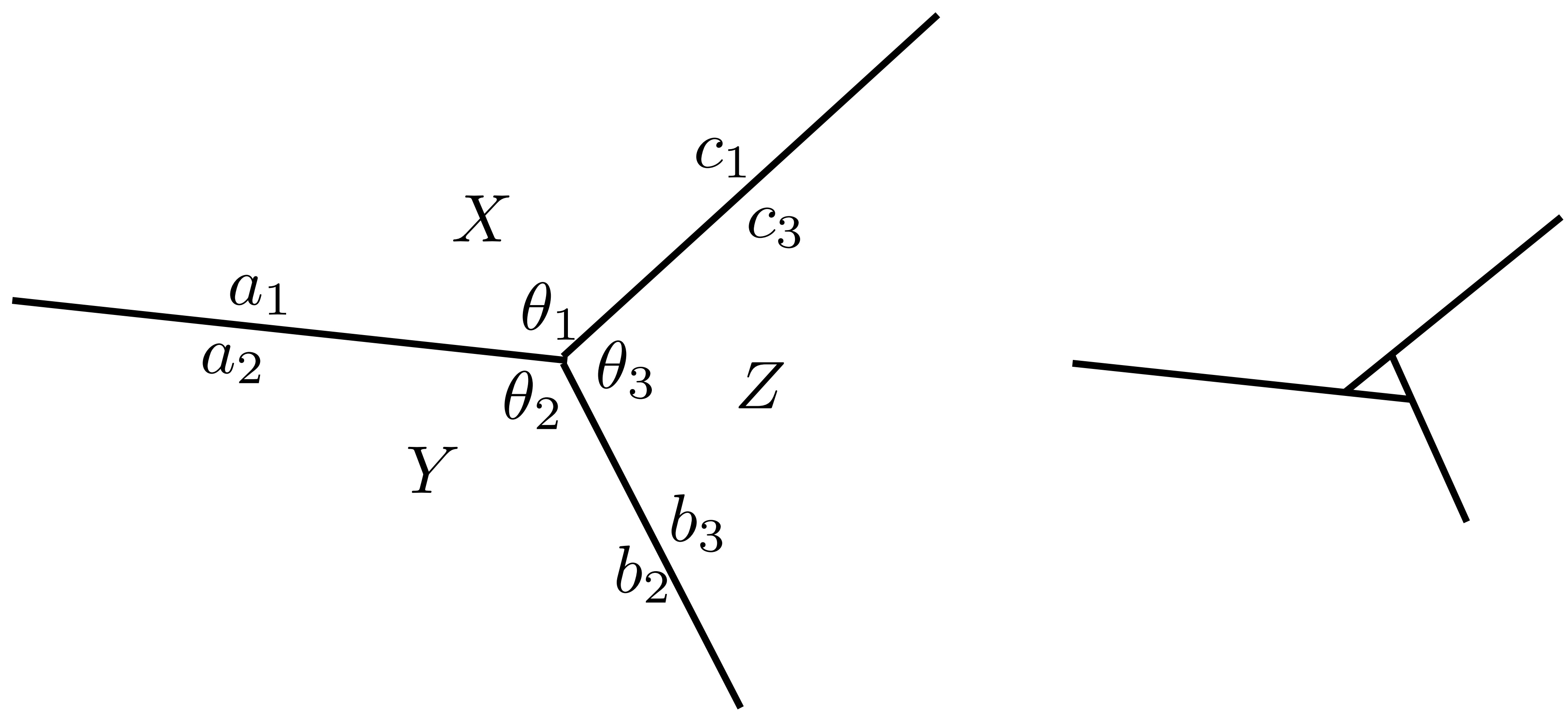


$$X = \frac{ac}{bd}$$



$$X = \frac{ace}{bdf}$$

At a degenerate vertex, biratios are defined by continuity:



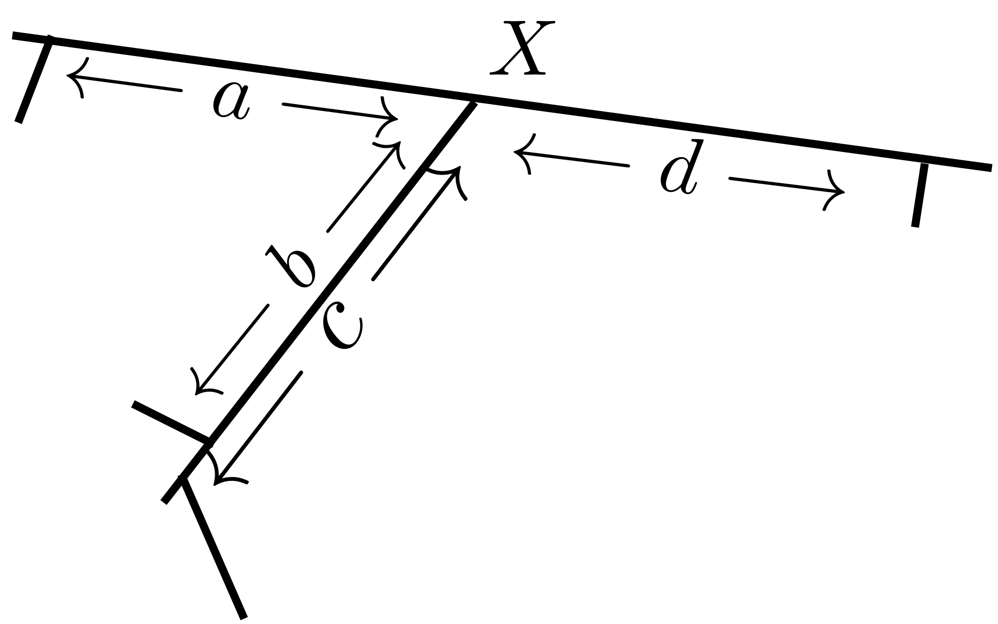
$$X = \frac{c_1 \sin\theta_3}{a_1 \sin\theta_2} \qquad Y = \frac{a_2 \sin\theta_1}{b_2 \sin\theta_3} \qquad Z = \frac{b_3 \sin\theta_2}{c_3 \sin\theta_1}$$

Proof idea: Let K be a Kasteleyn matrix with face weights X.

Find diagonal matrices D_W, D_B such that

$$\left.\begin{aligned} D_W K D_B \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} &= 0 \\ (1,\ldots,1) D_W K D_B &= 0 \end{aligned}\right\} \text{except on boundary.}$$

Use maximum principle to show embedding. $\square$

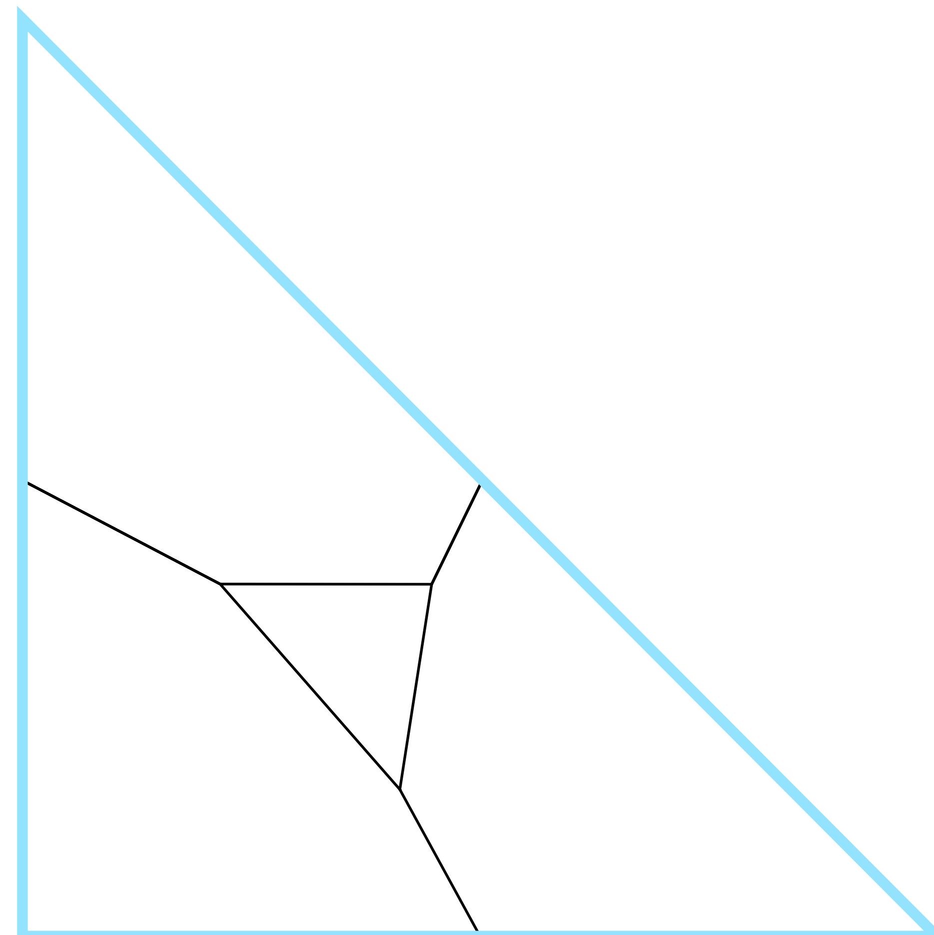


$$X = \frac{ac}{bd}$$

There are a number of *special cases* where one restricts the set of biratios.

Special case 1. Convex embeddings of graphs

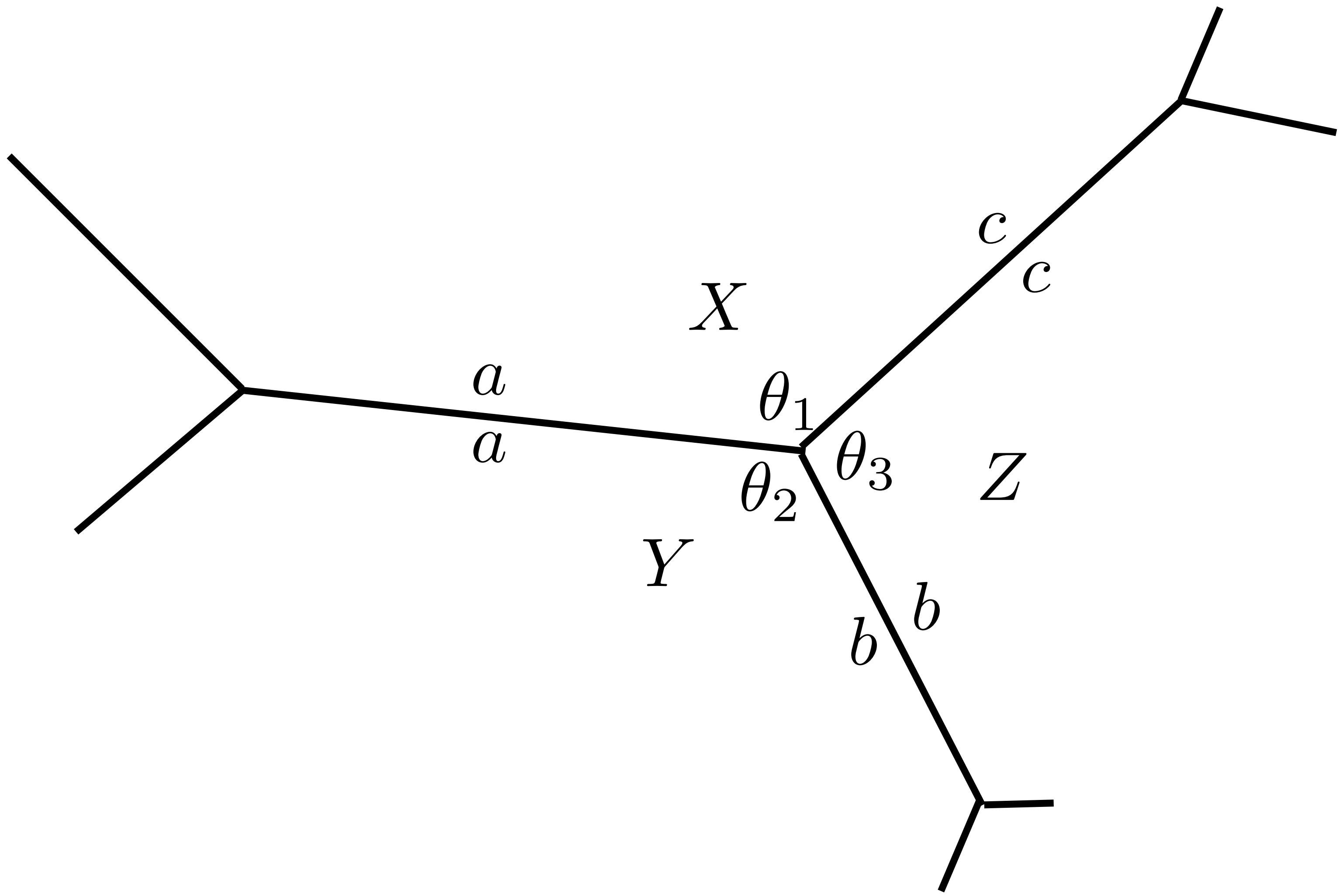
An embedding of a graph in $\mathbb{R}^2$ is *convex* if its faces are convex



Thm: The space of convex embeddings of G (with pinned boundary) is homeomorphic to $\mathbb{R}^{2V}$.

Proof: Take a nearby nondegenerate t-graph and set products of biratios around "vertices" to be 1.

Show that any such assignment of biratios results in an embedding. □



$$X = \frac{c \sin\theta_3}{a \sin\theta_2} \qquad Y = \frac{a \sin\theta_1}{b \sin\theta_3} \qquad Z = \frac{b \sin\theta_2}{c \sin\theta_1}$$

Note $XYZ = 1$

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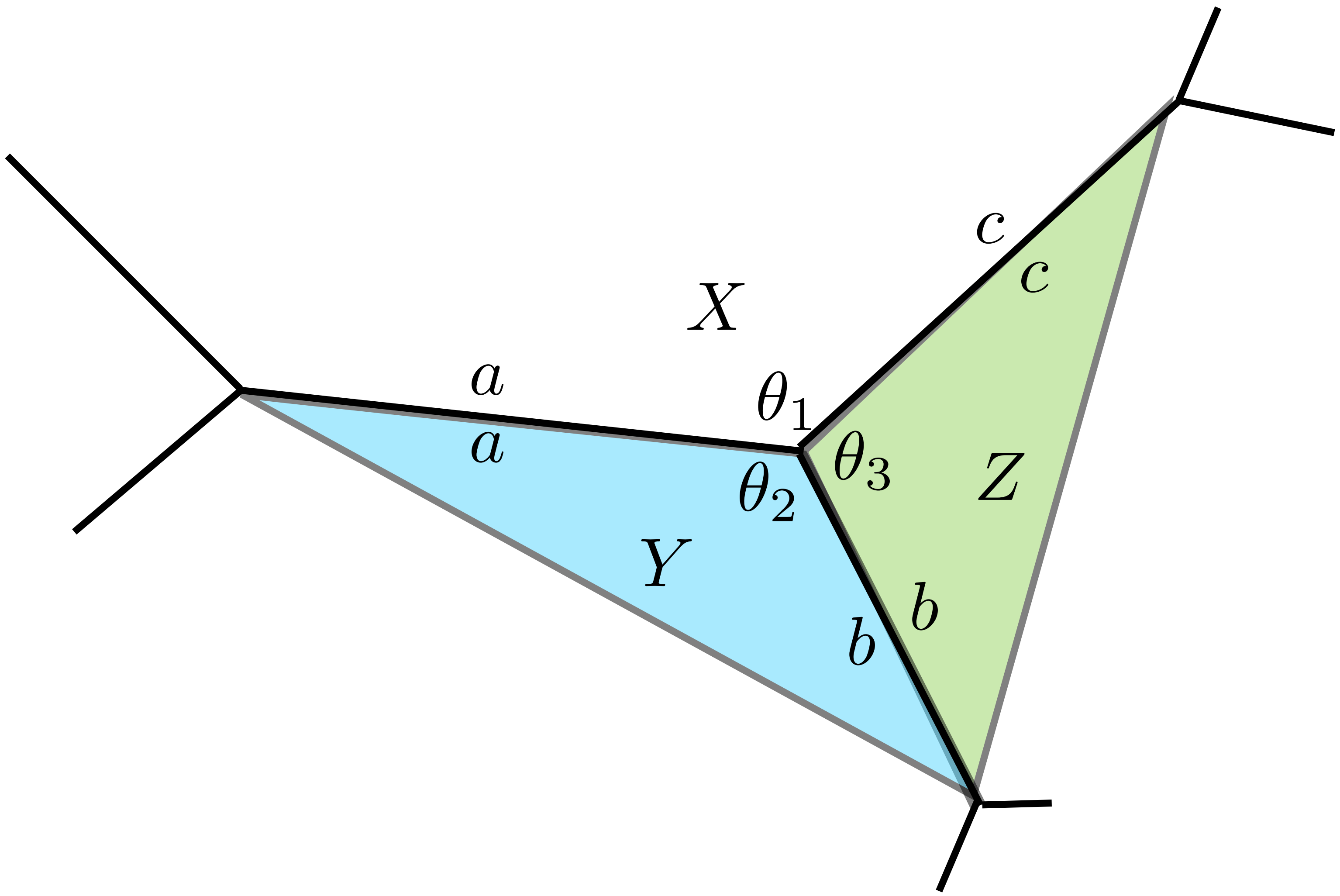

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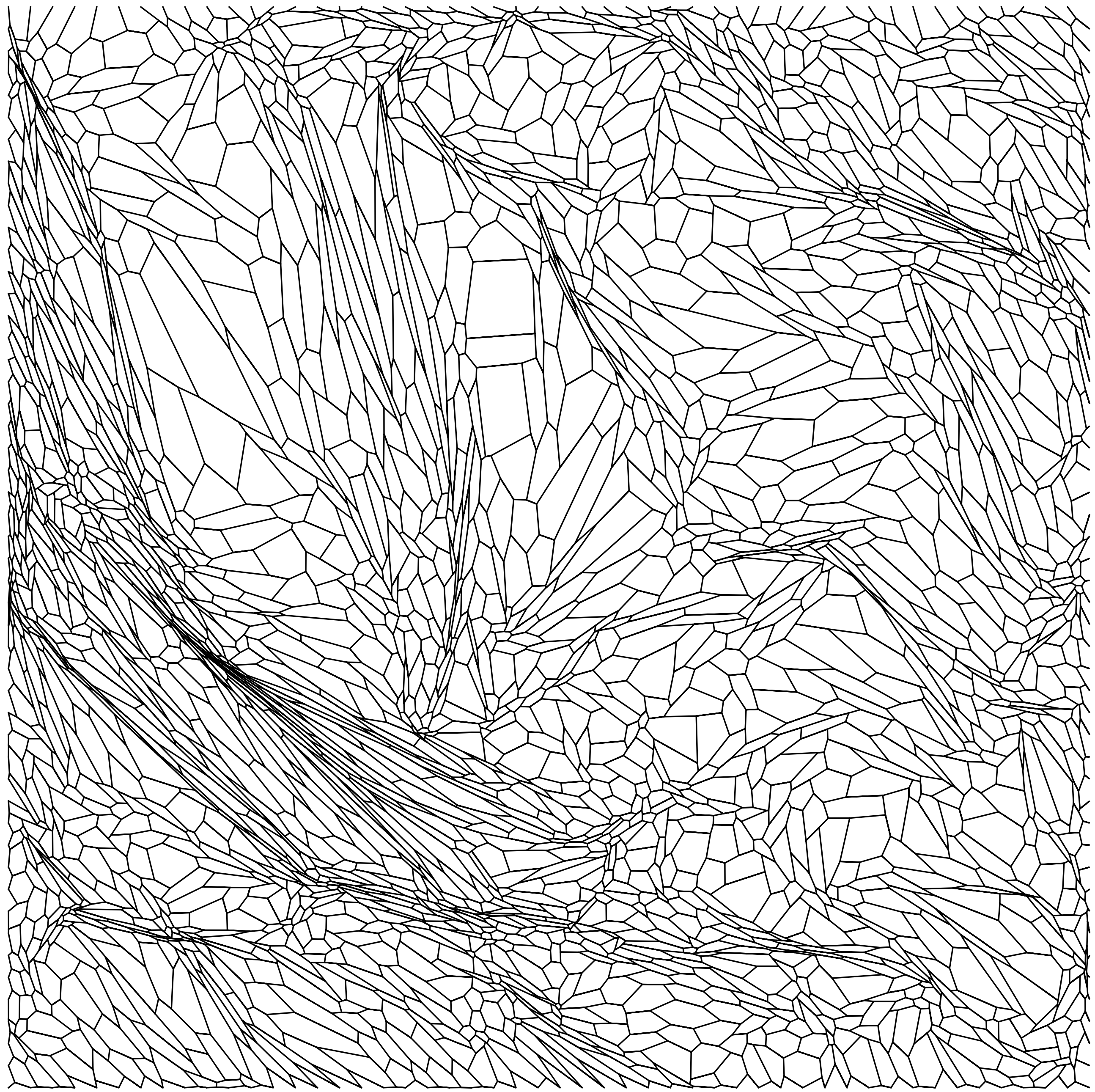
Note $XYZ = 1$

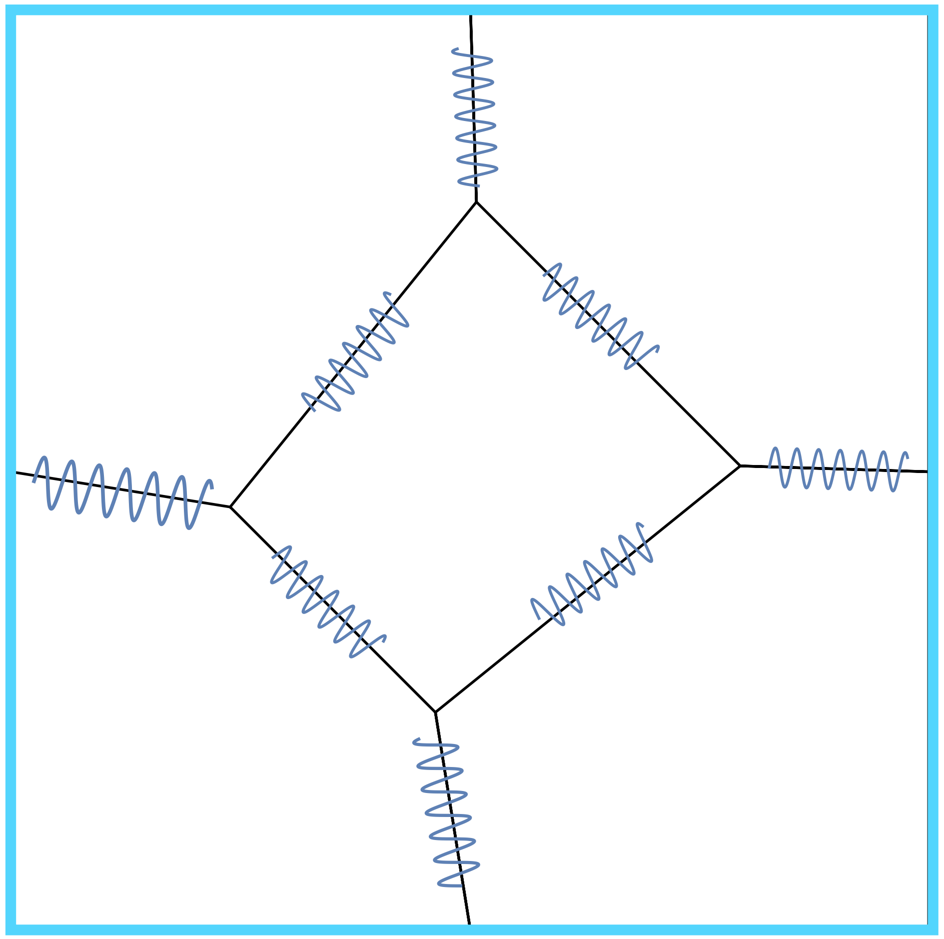
note that X, Y, Z are ratios of barycentric coordinates!



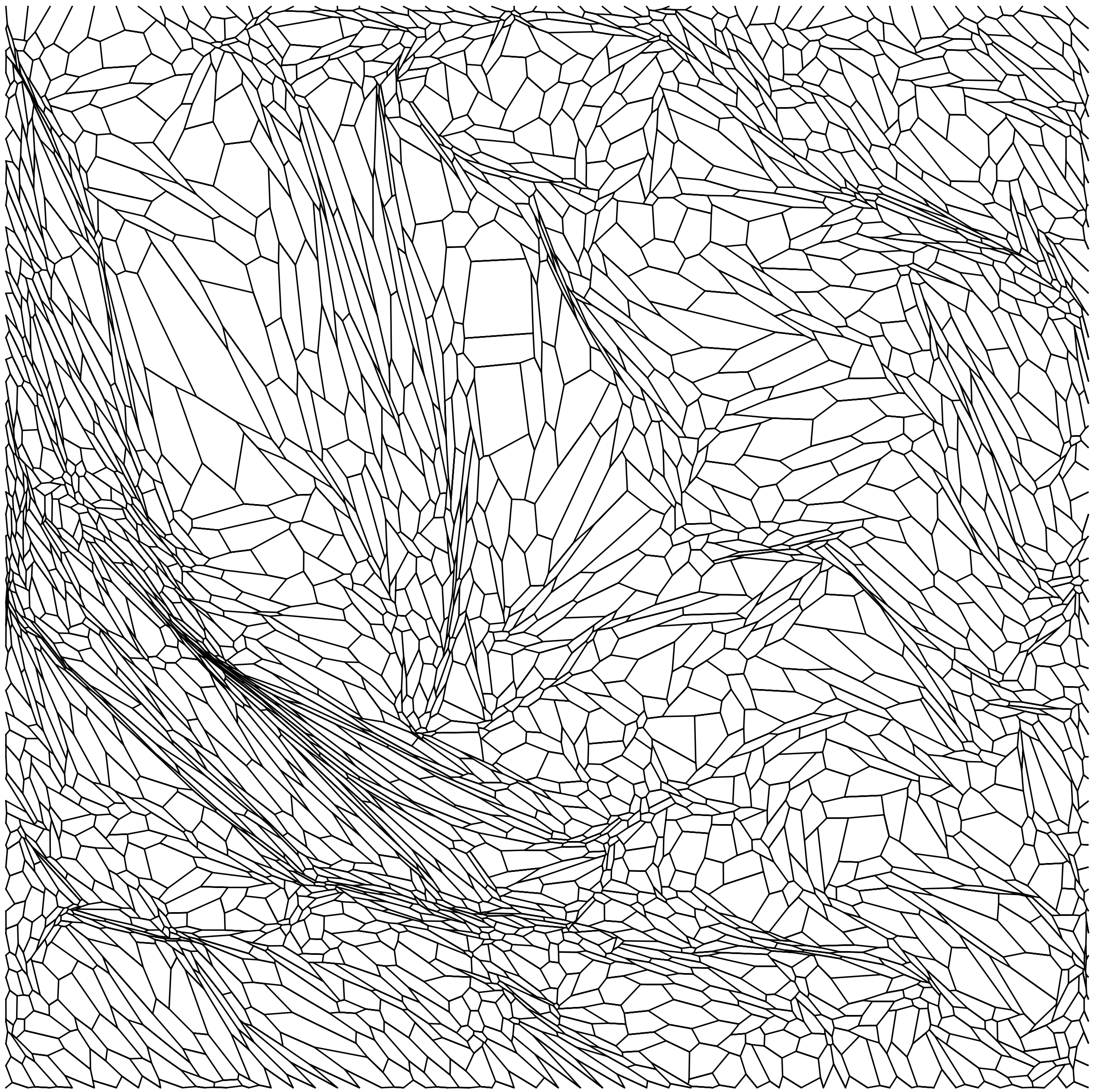
A natural probability measure on convex embeddings is obtained by choosing transition probabilities iid in $\{0 \leq p, q, p+q \leq 1\}$.

Special case 2.

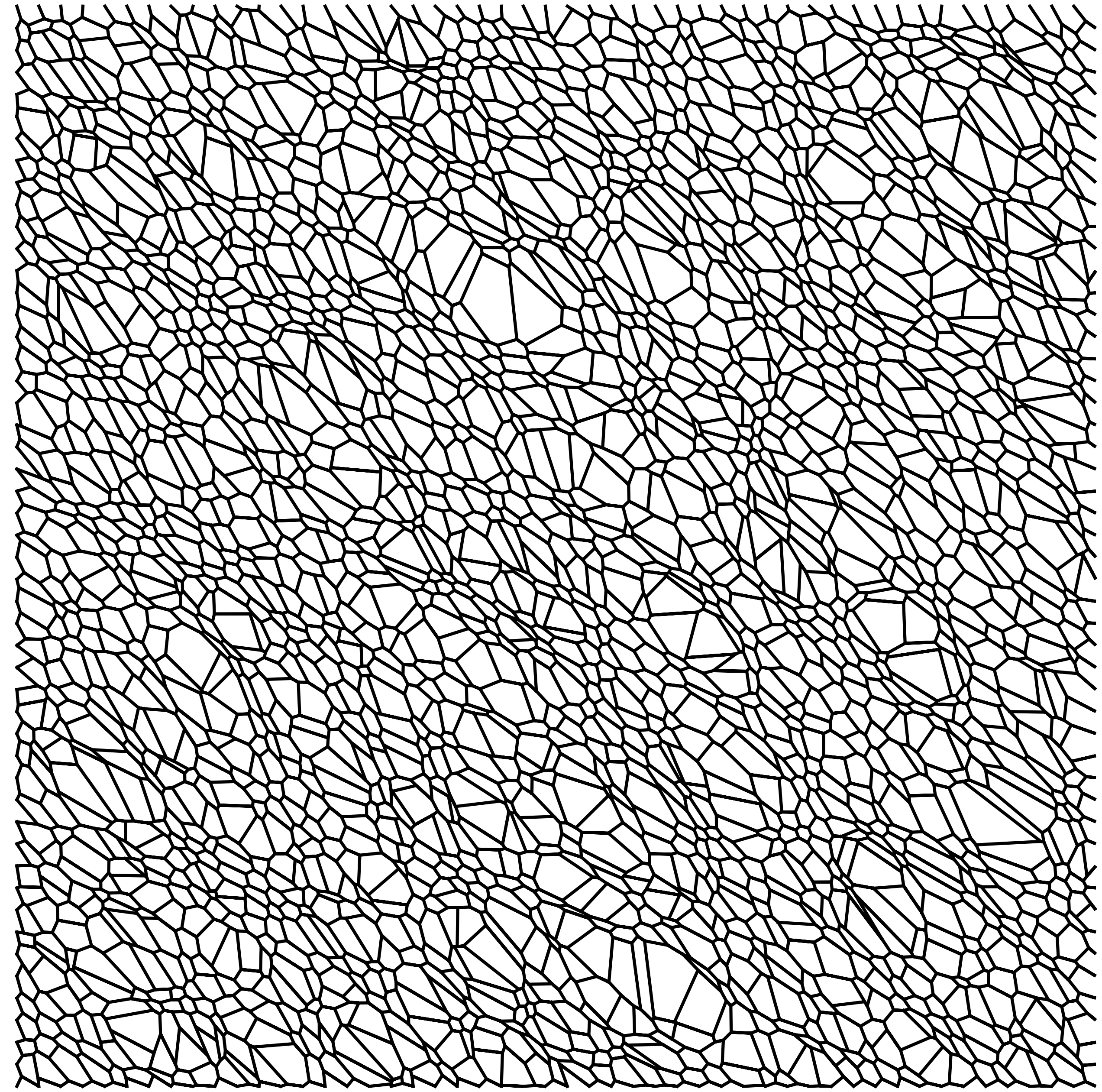
Product of Xs around both faces and vertices is 1.



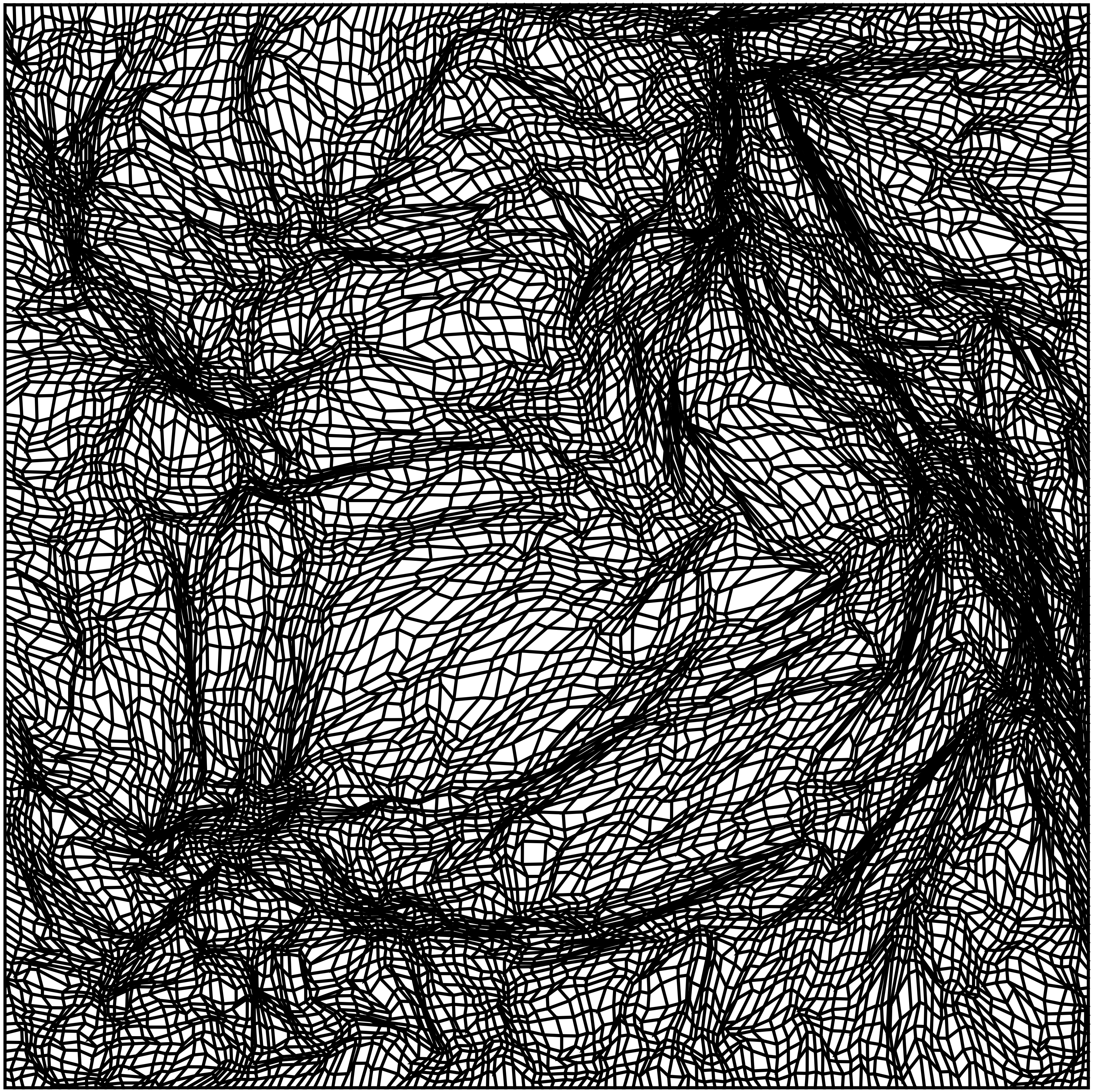
One can show that these conditions correspond to *harmonic embeddings* (spring networks / resistor networks)



Random convex embedding

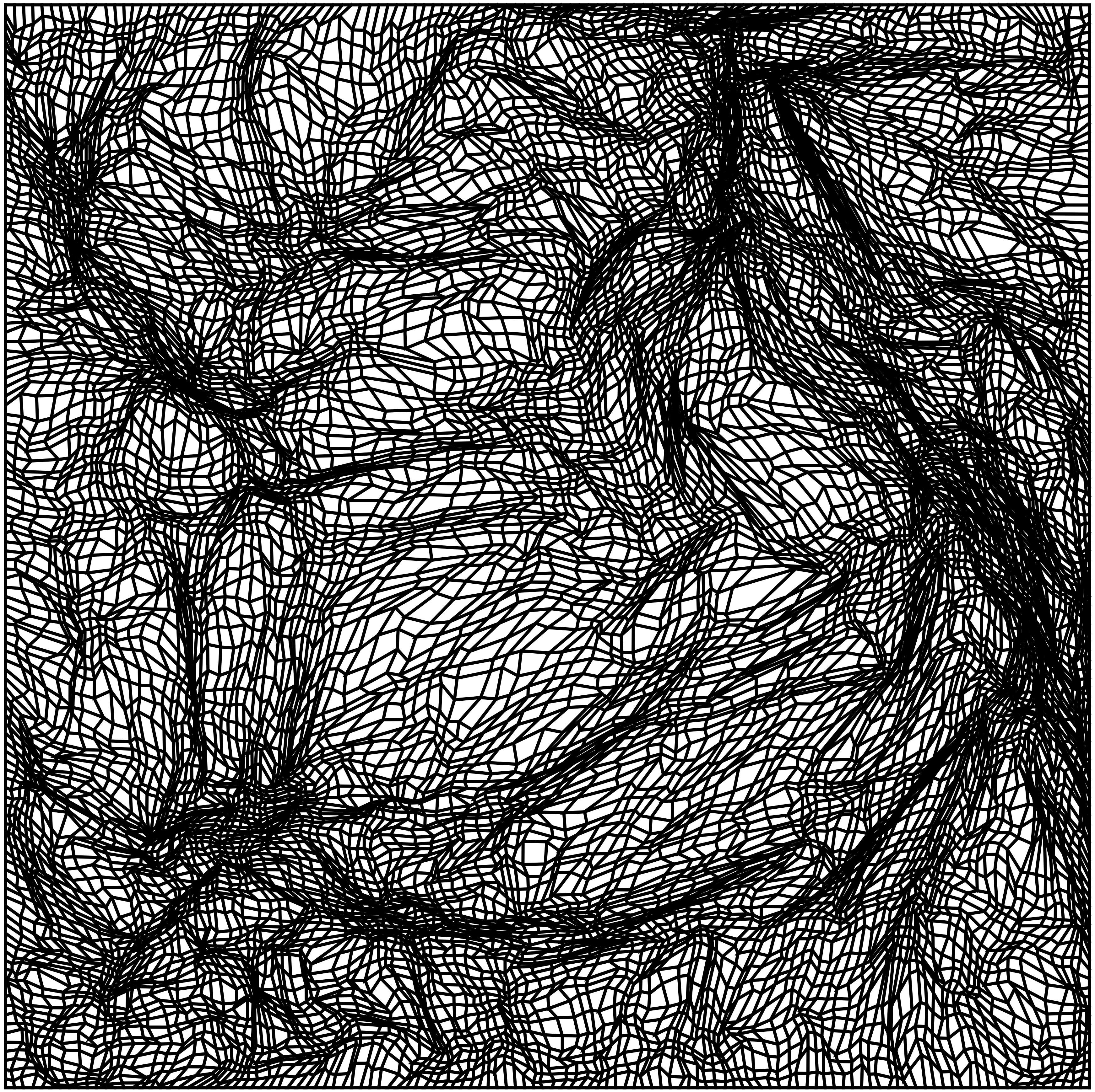


Random harmonic embedding



Conjecture: A random convex embedding does not have a scaling limit shape.

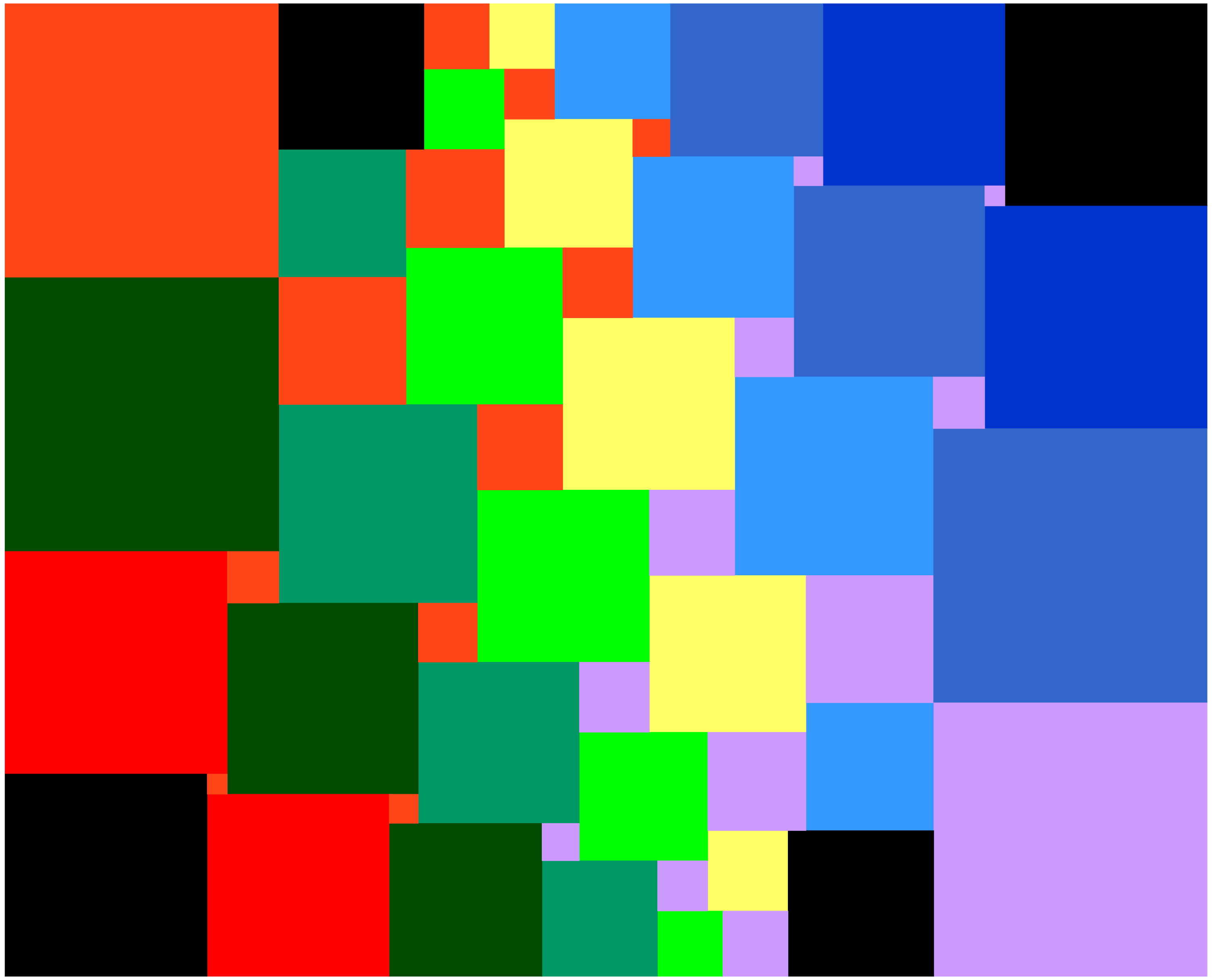
Conjecture [Zeitouni]: A random convex embedding has a scaling limit shape.
(would follow from CLT for RWRE)



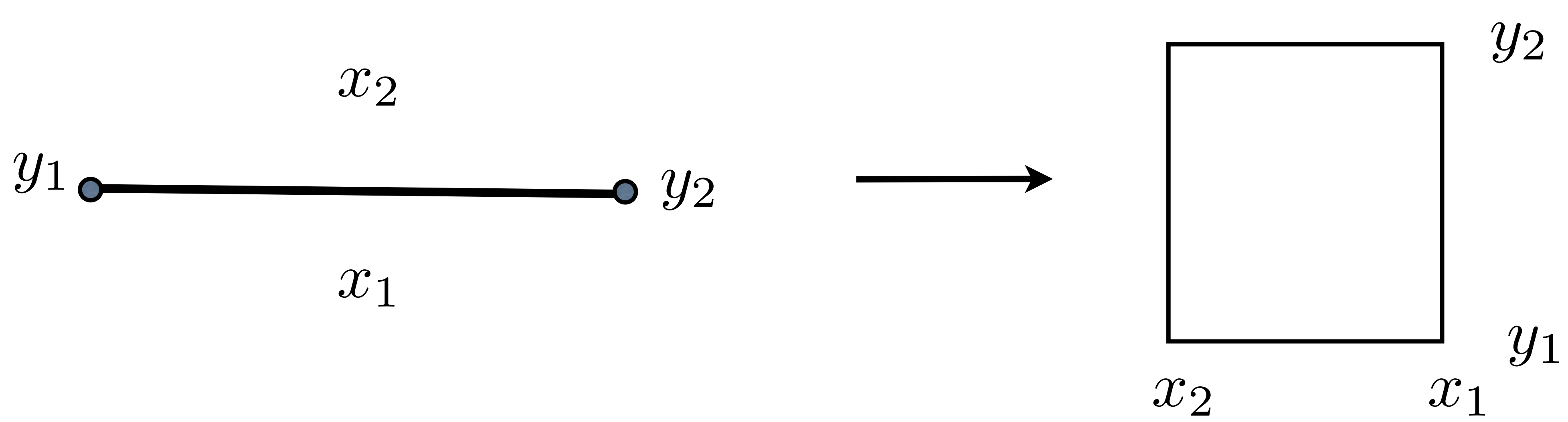
Q. Is there a natural probability measure on $\mathrm{Homeo}(\mathbb{D}^2, \mathbb{D}^2)$?

Special case 3. discrete analytic functions (Fix exact shapes up to scale)

e.g. square tilings (all Xs equal to 1)



Discrete analytic functions



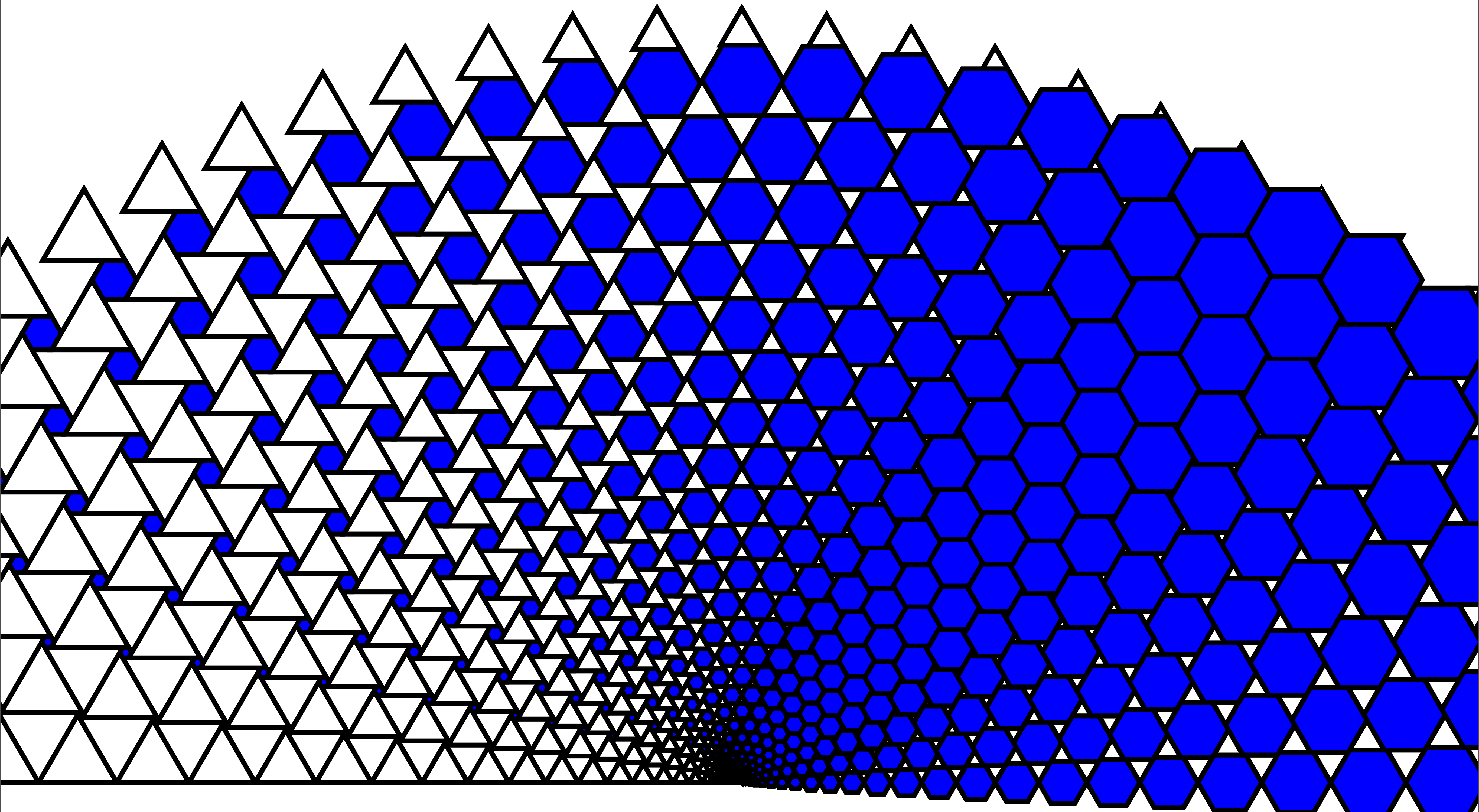
$$y_2 - y_1 = c(x_2 - x_1)$$

"discrete Cauchy-Riemann"

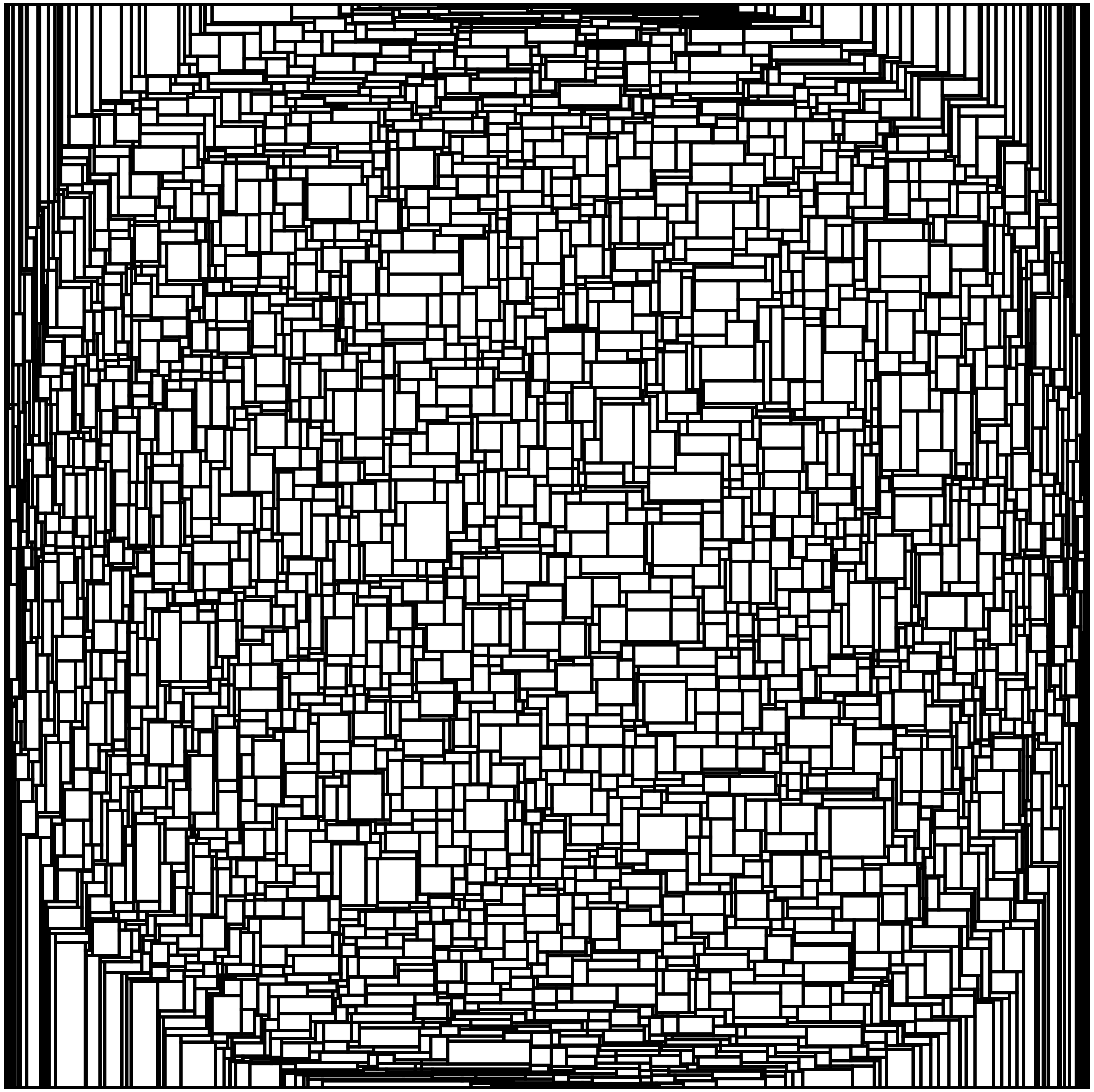
$$f_x = g_y$$
$$f_y = -g_x$$

More generally K is a discrete version of $\partial_{\bar{z}}$

e.g. regular hexagons and equilateral triangles (all X's equal to 1.)



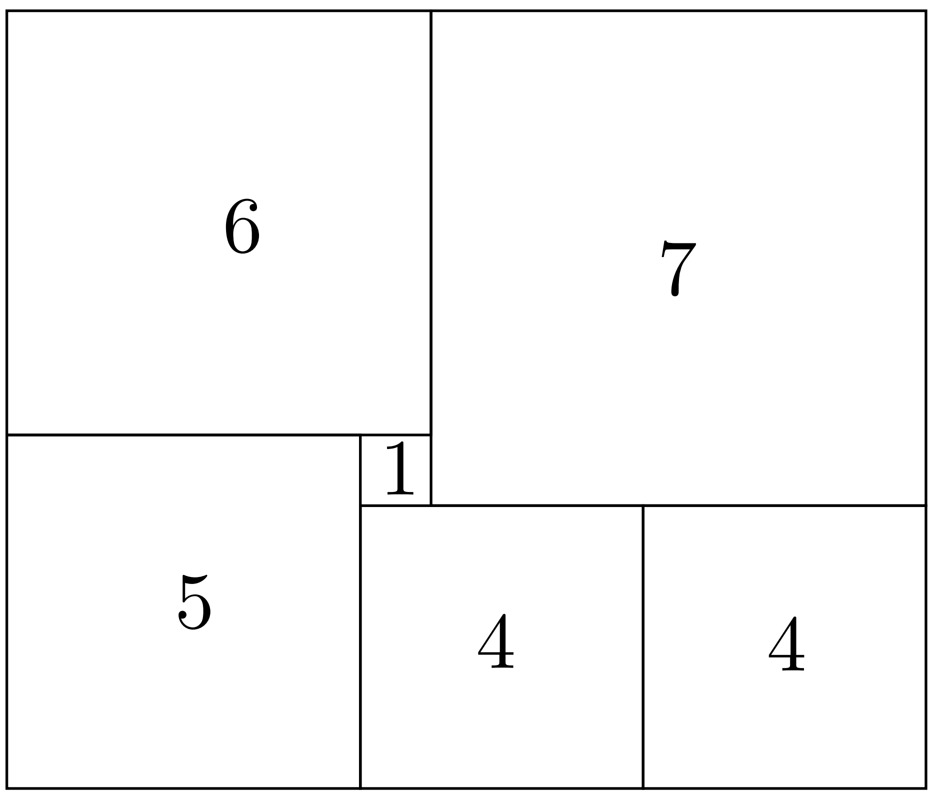
Rectangle tilings (product of adjacent Xs is 1)



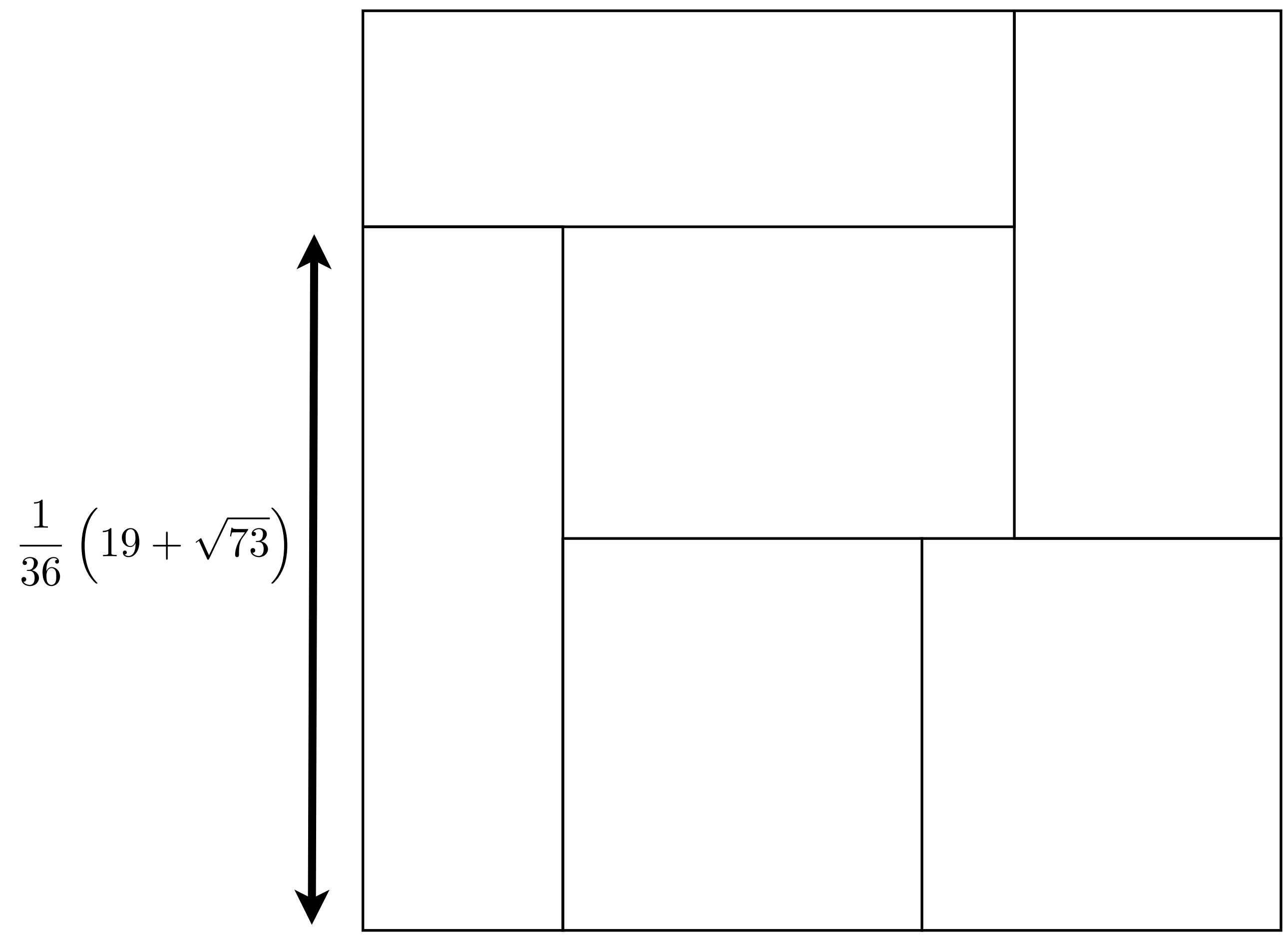
(square young tableau limit shape)

Fixed areas:

Given a rectangle tiling, there is an "isotopic" rectangle tiling with prescribed areas.



$x/y = 1$

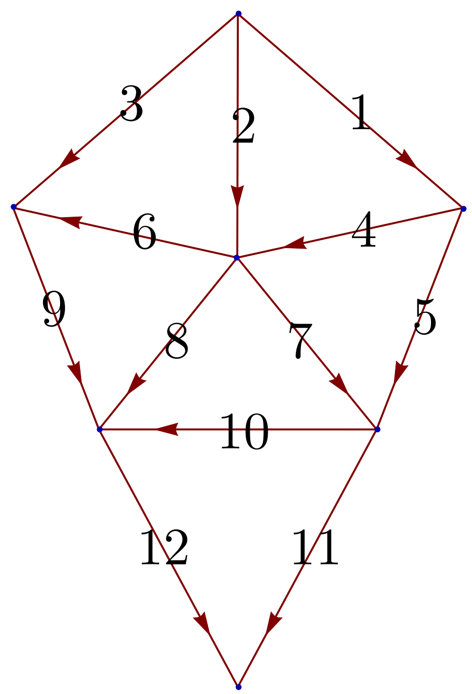


$xy = 1/6$

Thm [K-Abrams]

For every bipolar orientation of a planar graph, there is a unique Smith diagram with area-1 rectangles; that is, there is a unique choice of conductances so that the associated harmonic function has energies 1 and that orientation.

Bipolar orientation: Acyclic with exactly one source and sink (on outer boundary).



3
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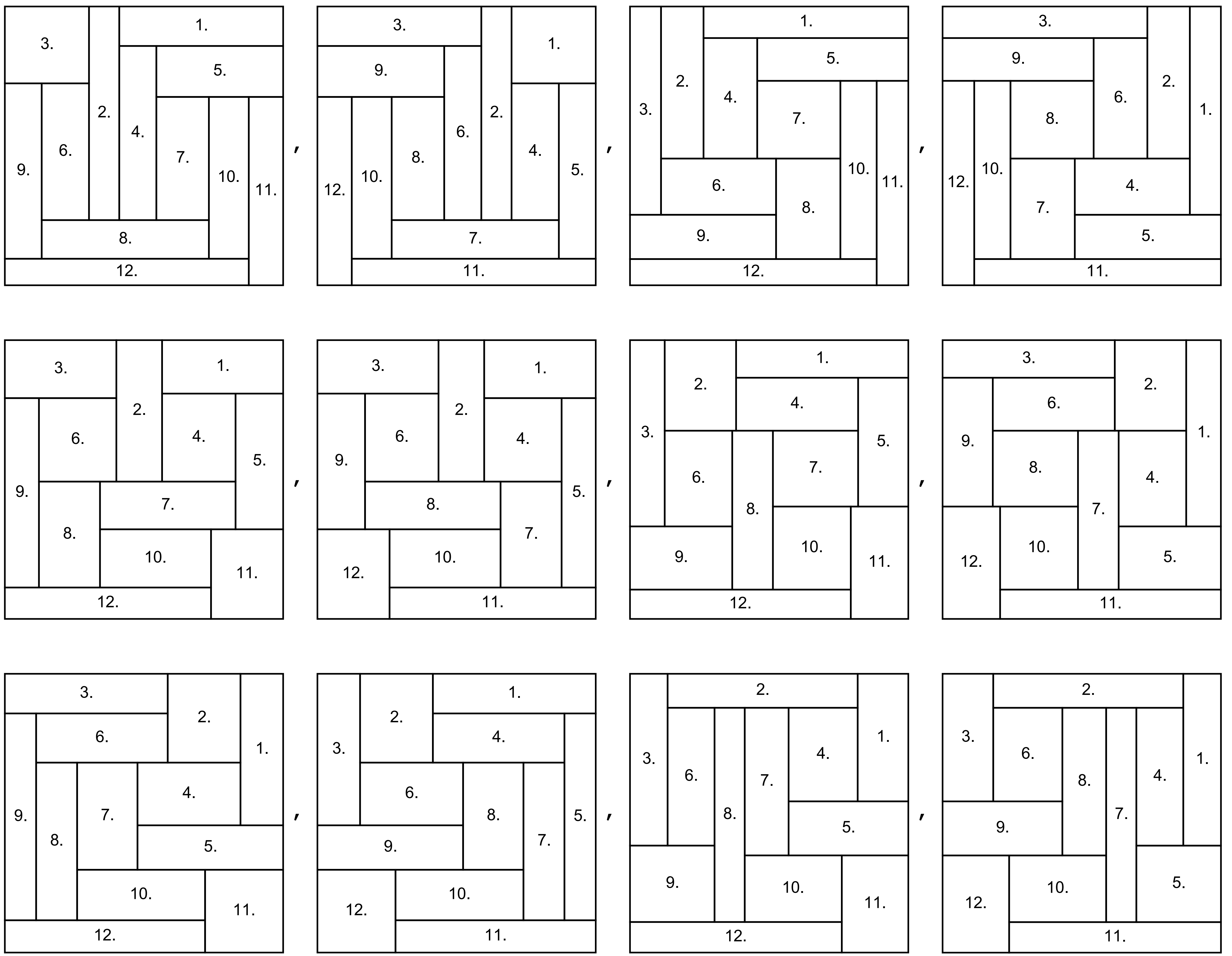
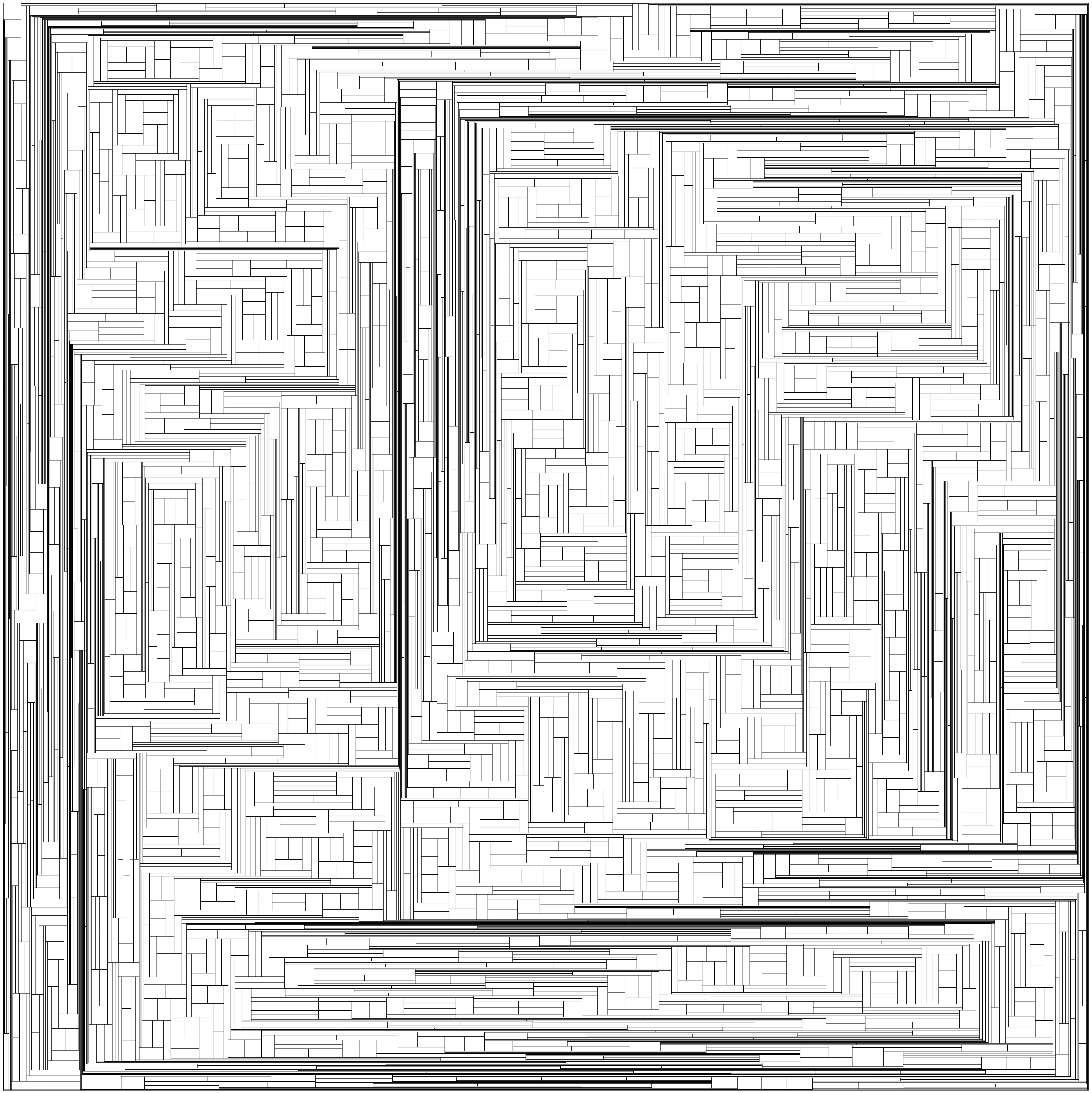
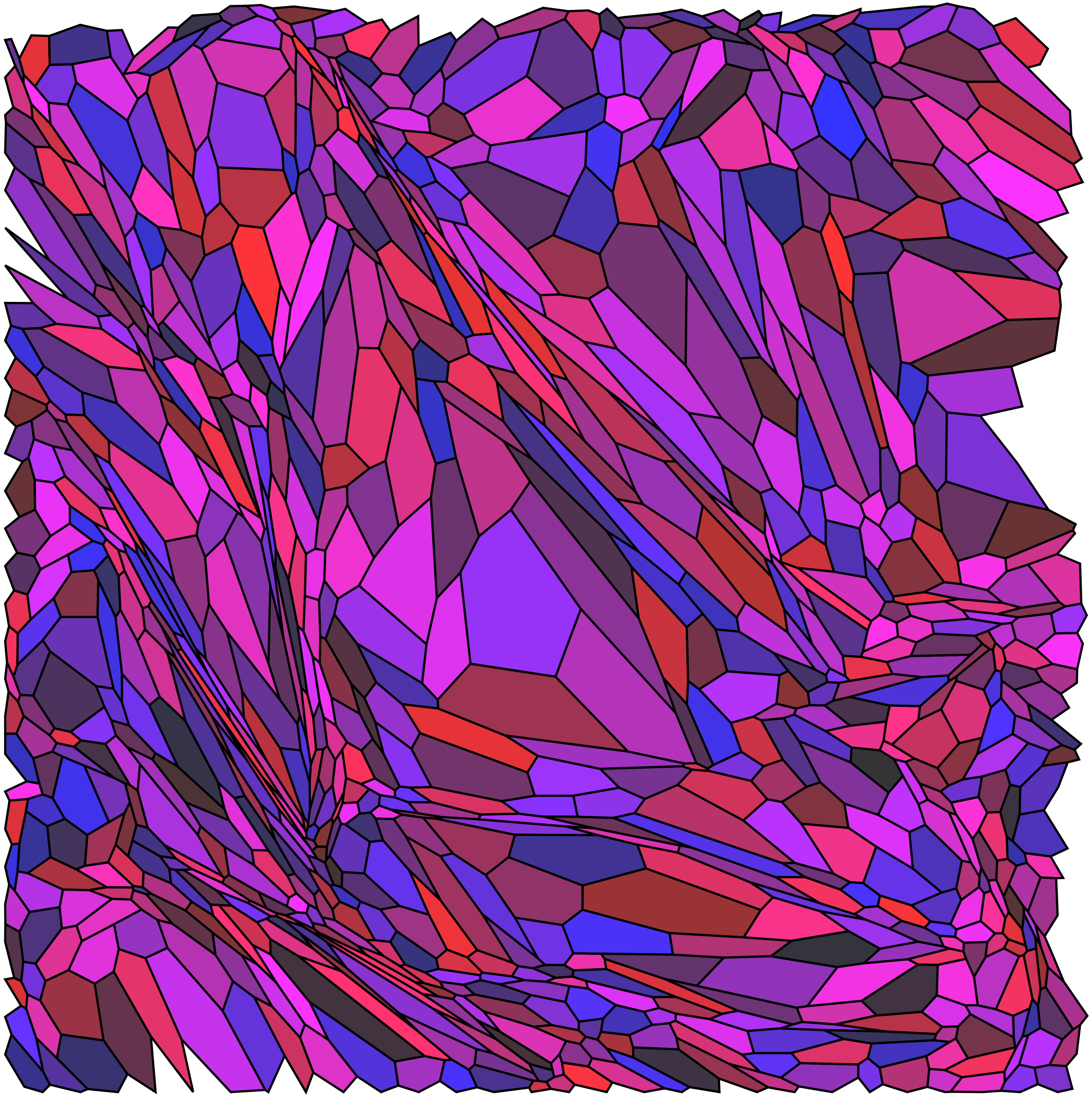
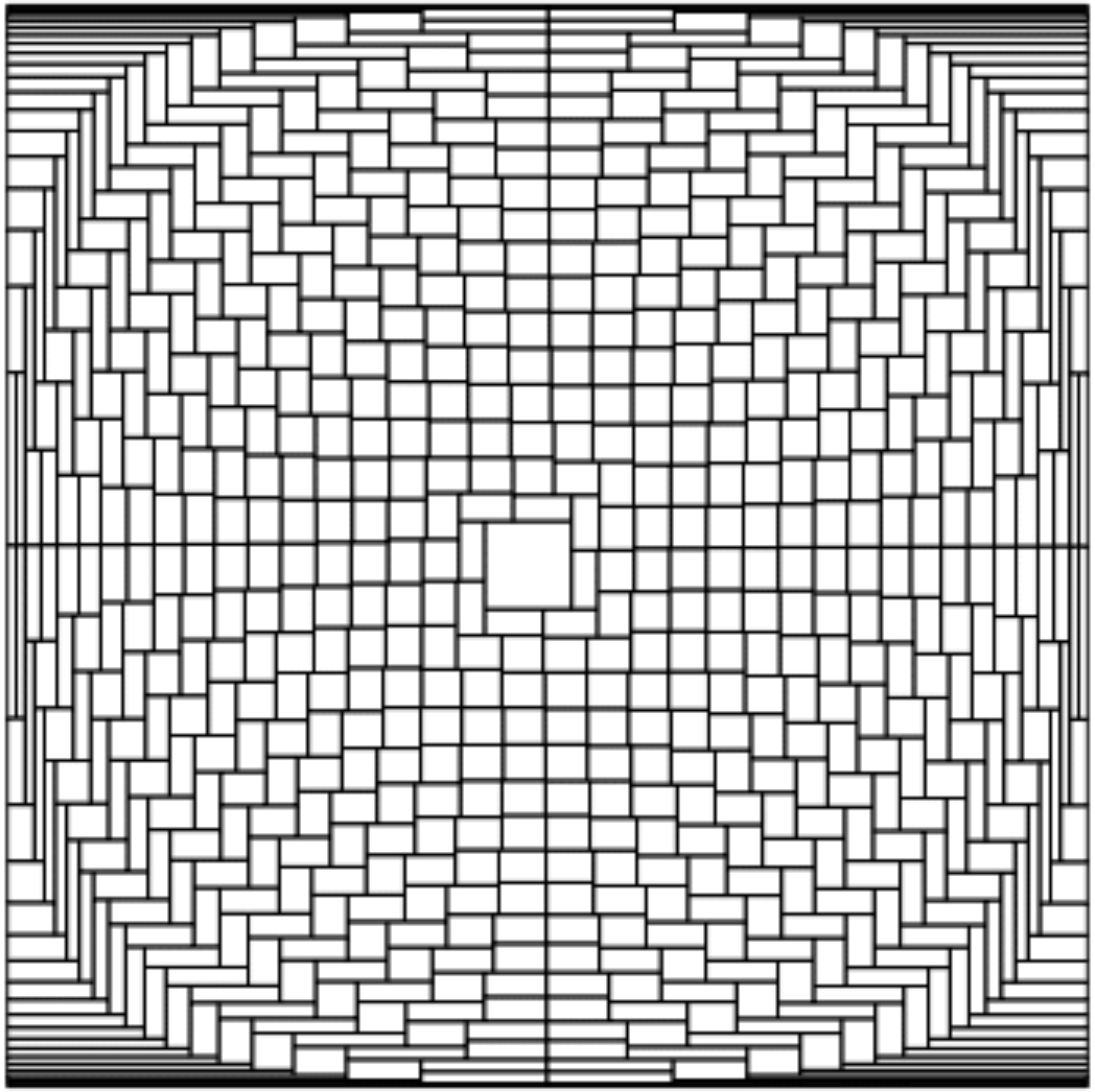


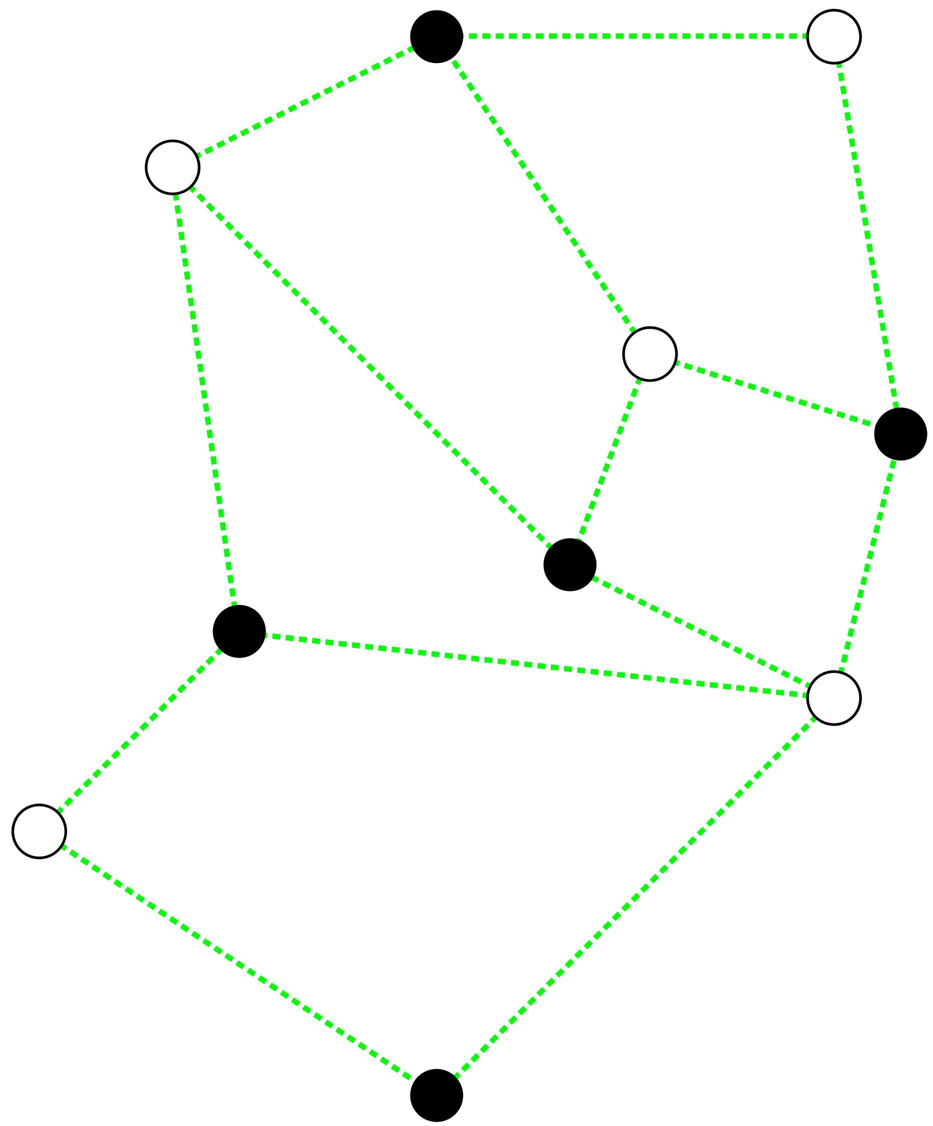
A random bipolar orientation of a random graph:

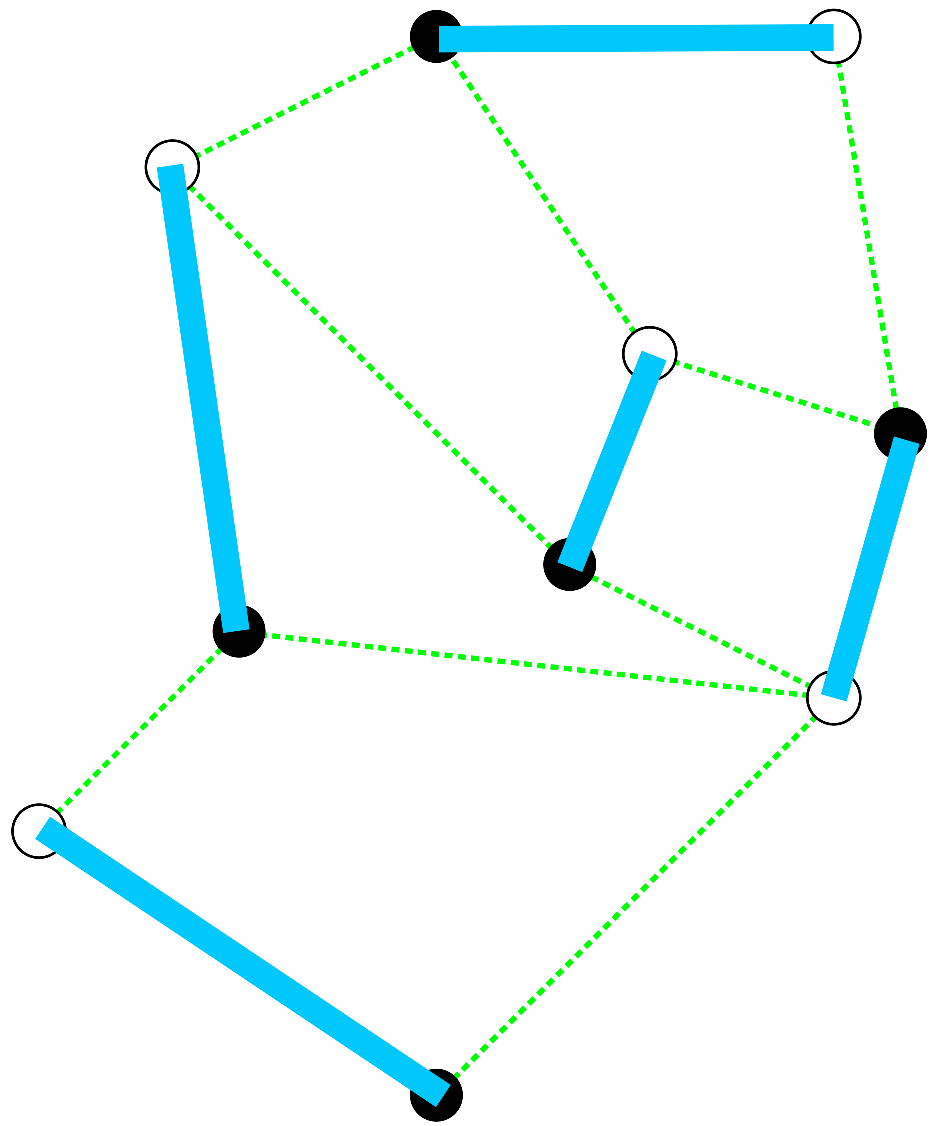


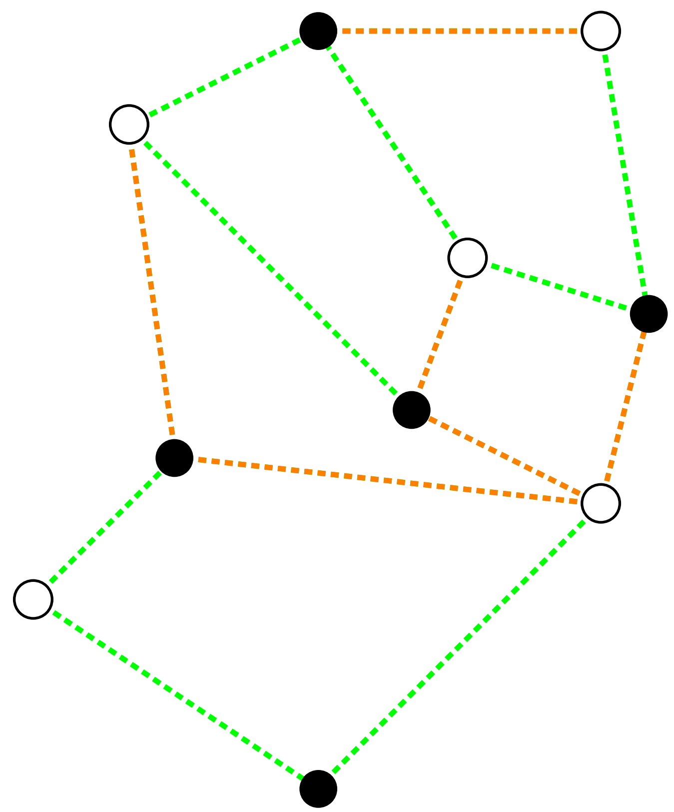


thank you for your attention!









$$K : \mathbb{R}^W \to \mathbb{R}^B$$

signed (weighted) adjacency matrix

Thm[Kasteleyn(1965)]:

$$\det K = \sum_{\text{dimer covers } m} wt(m).$$

Q. What is the geometry underlying K?